Appendix A

Derivation of equations

Derivation of equation (4.5)

Equations (4.4) and (4.5) are obtained by applying the variational method to equation (4.1). Here, I show how to calculate equation (4.5), defined in the stripe with in-plane anisotropy. The procedure to obtain equation (4.4) is similar. The energy per unit area for the stripe of width $L_2$ is:

$$ G_{L_2} = \int_0^{L_2} \left( A \left( \frac{d\theta}{dx} \right)^2 + k_{sh} + k_{eff} \sin^2 \theta \right) dx $$  \hspace{1cm} (A.1)

This functional can be minimized by considering an arbitrary small variation $\delta \theta(x)$. In this way the variation of the energy results:

$$ \delta G_{L_2} = 2 \int_0^{L_2} \left( A \frac{d}{dx} \frac{d\theta}{dx} + k_{eff} \sin \theta \cos \theta \right) \delta \theta dx $$  \hspace{1cm} (A.2)

Integration by parts brings to:

$$ \delta G_{L_2} = 2 \int_0^{L_2} \left( -A \frac{d^2\theta}{dx^2} + k_{eff} \sin \theta \cos \theta \right) \delta \theta dx $$  \hspace{1cm} (A.3)

where the condition of symmetry, that gives $\theta(0) = \theta(L_2)$, has been used. The extreme of equation (A.1) is obtained if $\delta G_{L_2} = 0$ for each arbitrary variation $\delta \theta(x)$. This condition brings to the Euler-Lagrange equation of the problem that is given by:

$$ \lambda_2^2 \frac{d^2}{dx^2} \frac{d\theta}{dx} + \sin \theta \cos \theta = 0 $$  \hspace{1cm} (A.4)

with $\lambda_2 = \sqrt{-\frac{A}{k_{eff}}} > 0$. By multiplying for $\frac{d\theta}{dx}$ and integrating by parts, we obtain equation (4.5):
Appendix A - Derivation of equations

\[
\left( \frac{d\theta}{dx} \right)^2 = -\frac{\sin^2 \theta}{\lambda^2} + c_2 \tag{A.5}
\]

### Elliptical integral of the first kind

\[
F(\varphi|m) = \int_0^\varphi \frac{1}{\sqrt{1 - m \sin^2 \theta}} d\theta \tag{A.6}
\]

### Elliptical integral of the second kind

\[
E(\varphi|m) = \int_0^\varphi \sqrt{1 - m \sin^2 \theta} d\theta \tag{A.7}
\]

### Definitions

Sine and cosine amplitude:

\[
\sin(F(\varphi|m)) = \sin \varphi \quad \cos(F(\varphi|m)) = \cos \varphi \tag{A.8}
\]

### Useful properties

Introducing the notation \( \xi = F(\varphi|m) \) for the elliptical integral of the first kind we have:

\[
\sin^2 \xi + \cos^2 \eta = 1 \tag{A.9}
\]

\[
F(\alpha|m) = \frac{F(\beta|m^{-1})}{\sqrt{m}} \tag{A.10}
\]

with \( \sin \alpha = \sin \beta / \sqrt{m} \)

\[
\frac{d\sin \xi}{du} = \sqrt{(1 - \sin^2 \xi)(1 - m\sin^2 \xi)} \tag{A.11}
\]

\[
\int \sin^2 \xi d\xi = \frac{\xi - E(\varphi|m)}{m} \tag{A.12}
\]

\[
\int \cos^2 \xi d\xi = \frac{E(\varphi|m) - (1 - m)\xi}{m} \tag{A.13}
\]

### Derivation of equations (4.10) and (4.11)

\[
F(\theta_x|m) - F(\theta_0|m) = \frac{x}{\lambda^2 \sqrt{m}} \tag{A.14}
\]
with $m = 1/\sin^2 \theta_{L_2}$. By using the relation (A.10) equation (A.14) becomes:

$$F(\phi_x|m^{-1}) - F(\phi_0|m^{-1}) = \frac{x}{\lambda_2}$$  \hfill (A.15)

with

$$\sin \phi_x = \sqrt{m \sin \theta_x} \quad \sin \phi_0 = \sqrt{m \sin \theta_0}$$  \hfill (A.16)

The system of equations (A.15) and (A.16) can be compacted in the following manner by inserting relations (A.16) in equation (A.15):

$$F(\arcsin \left| \frac{\sin \theta_x}{\sin \theta_{L_2}} \right| \sin^2 \theta_{L_2}) - F(\arcsin \left| \frac{\sin \theta_0}{\sin \theta_{L_2}} \right| \sin^2 \theta_{L_2}) = \frac{x}{\lambda_2}$$  \hfill (A.17)

that is equation (4.10). The analogous of equation (A.14) for the case $-L_1 \leq x \leq 0$ can be deduced from equation (4.6) and results:

$$F(\pi/2 - \theta_x|m) - F(\pi/2 - \theta_0|m) = -\frac{x}{\lambda_1 \sqrt{m}}$$  \hfill (A.18)

with $m = 1/\cos^2 \theta_{L_1}$. By using the relation (A.10) equations (A.18) becomes:

$$F(\phi_x|m^{-1}) - F(\phi_0|m^{-1}) = -\frac{x}{\lambda_1}$$  \hfill (A.19)

with

$$\sin \phi_x = \sqrt{m \cos \theta_x} \quad \sin \phi_0 = \sqrt{m \cos \theta_0}$$  \hfill (A.20)

Substituting relations (A.20) in equation (A.19) we obtain equation (4.11):

$$F(\arcsin \left| \frac{\cos \theta_x}{\cos \theta_{L_1}} \right| \cos^2 \theta_{L_1}) - F(\arcsin \left| \frac{\cos \theta_0}{\cos \theta_{L_1}} \right| \cos^2 \theta_{L_1}) = -\frac{x}{\lambda_1}$$  \hfill (A.21)

**Derivation of equation (4.17)**

The following integrals contained in equation (4.16) have to be solved:

$$I_1 = \int_{\theta_{L_1}}^{0} \left( \frac{d\theta_x}{dx} \right)^2 dx$$  \hfill (A.22)

$$I_2 = \int_{0}^{\theta_{L_2}} \left( \frac{d\theta_x}{dx} \right)^2 dx$$  \hfill (A.23)

Equation (A.16) can be written as:
Appendix A - Derivation of equations

\[ \theta_x = \arcsin(\sqrt{m_2} \sin \phi_x) \]  \hspace{1cm} (A.24)

with \( m_2 = \sin^2 \theta \frac{L_2}{\lambda_2} \). By using relations (A.8), (A.9) and (A.11) the derivative of equation (A.24) becomes:

\[ \frac{d\theta_x}{dx} = \frac{\sqrt{m_2}}{\lambda_2} \text{cn}(F(\phi_x|m_2)) \]  \hspace{1cm} (A.25)

Substituting in integral (A.23) we obtain:

\[ I_2 = \frac{m_2}{\lambda_2} \int_0^{L_2} \text{cn}^2 z dz \]  \hspace{1cm} (A.26)

where \( z = \frac{x}{\lambda_2} + F(\phi_0|m_2) \) and equation (A.15) has been used. Using relation (A.13) integral (A.26) becomes:

\[ I_2 = \frac{E(\phi \frac{L_2}{\lambda_2}|m_2) - E(\phi_0|m_2) - (1 - m_2) \frac{L_2}{2\lambda_2}}{\lambda_2} \]  \hspace{1cm} (A.27)

From equation (A.22) in the same way we obtain:

\[ I_1 = \frac{E(\phi \frac{L_1}{2\lambda_1}|m_1) - E(\phi_0|m_1) - (1 - m_1) \frac{L_1}{2\lambda_1}}{\lambda_1} \]  \hspace{1cm} (A.28)

with \( m_1 = \cos^2 \theta \frac{-L_1}{\lambda_2} \). Finally, substituting integrals (A.27) and (A.28) in equation (4.16) we obtain equation (4.17).

**Derivation of equation (4.20)**

To calculate equation (4.20) we need to solve the following integrals:

\[ I_1 = \int_{-\frac{L_1}{2}}^{0} \sin^2 \theta_x dx \]  \hspace{1cm} (A.29)

\[ I_2 = \int_{0}^{\frac{L_2}{2}} \sin^2 \theta_x dx \]  \hspace{1cm} (A.30)

Using relation (A.20) equation (A.29) becomes:

\[ I_1 = \frac{L_1}{2} - m_1 I_1^* \]  \hspace{1cm} (A.31)

with \( m_1 = \cos^2 \theta \frac{-L_1}{\lambda_2} \) and:

\[ I_1^* = \int_{-\frac{L_1}{2}}^{0} \sin^2 \phi_x dx \]  \hspace{1cm} (A.32)

Considering equations (A.8) and (A.19) integral (A.32) becomes:
\[ I_1^* = \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} \text{sn}^2 \left( -\frac{x}{\lambda_1} + F(\phi_x|m_1) \right) dx \quad (A.33) \]

After the change of variable \( z = -\frac{x}{\lambda_1} + F(\phi_x|m_1) \), using relation (A.12) we obtain:

\[ I_1^* = \frac{L_1}{2m_1} - \frac{\lambda_1}{m_1} (E(\phi_x|\frac{L_1}{2}) - E(\phi_0|m_1)) \quad (A.34) \]

Substituting in (A.31) we have:

\[ I_1 = \lambda_1 (E(\phi_x|\frac{L_1}{2}) - E(\phi_0|m_1)) \quad (A.35) \]

In a similar way we calculate integral \( I_2 \) using equations (A.15) and (A.16):

\[ I_2 = \int_{0}^{\frac{L_2}{2}} \text{sn}^2 \left( \frac{x}{\lambda_2} + F(\phi_x|m_2) \right) dx \quad (A.36) \]

with \( m_2 = \sin^2 \theta \frac{L_2}{2} \). Using the new variable \( z = \frac{x}{\lambda_2} + F(\phi_x|m_2) \) and equation (A.12) we obtain:

\[ I_2 = \frac{L_2}{2} - \lambda_2 (E(\phi_x|\frac{L_2}{2}) - E(\phi_0|m_2)) \quad (A.37) \]

Finally, inserting integrals (A.36) and (A.37) in equation (4.19) we obtain equation (4.20).