Chapter 4

Hedging in Complete and Incomplete Markets

The problem of pricing and hedging a contingent claim with payoff $H$ is well understood in the context of arbitrage-free option pricing in complete markets (see Black and Scholes [4], Merton [30]). In this situation, a perfect hedge is always possible, i.e., there exists a dynamic strategy such that trading in the underlying assets replicates the payoff of the contingent claim. Then, the price of the contingent claim turns out to be the expectation of $H$ with respect to the equivalent martingale measure which is unique. However, the possibility of a perfect hedge is restricted to the complete market and thus, to certain models and restrictive assumptions. In more realistic models the market will be incomplete, i.e., a perfect hedge as in the Black-Scholes-Merton model is not possible and the equivalent martingale measure is not unique any longer. Thus, a contingent claim bears an intrinsic risk that cannot be hedged away completely. Therefore, we are faced with the problem of searching strategies which reduce the risk of the resulting shortfall as much as possible.

One can still stay on the safe side using a superhedging strategy (see [13] for a survey). Then, the replicating portfolio at final time $T$ is in any case larger than the payoff of the contingent claim. But from a practical point of view, the cost of superhedging is often too high (see for instance [21]). For this reason, we consider the possibility of investing less capital than the superhedging price of the liability. This leads to a shortfall, the risk of which, measured by a suitable risk measure, should be minimized.

A similar problem arises when hedging in complete markets and the investor is unwilling or unable to pay the unique arbitrage free price of a contingent claim and wants to invest a sum less than this price. The aim is to find a hedging strategy that minimizes the losses due to the difference between the claim and the hedging portfolio at time $T$, measured by a suitable risk measure. This is a special case of the above mentioned problem since we have only to deal with an unique equivalent martingale measure. In Section 4.1, we shall consider this problem and the problem
of hedging in special incomplete markets. In Section 4.2, we consider the general incomplete market.

We study the hedging problem using different types of risk measures. First, we give a general result and then deduce in the following subsections the corresponding results for different kinds of risk measures and compare the obtained results with the recent literature.

In our setting, the discounted price process of the underlying assets is described as an $\mathbb{R}^d$-valued semimartingale $S = (S_t)_{t \in [0,T]}$ on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t \in [0,T]}$ and $\mathcal{F} = \mathcal{F}_T$. A semimartingale is the sum of a continuous local martingale and a finite-variation process that is right-continuous with left-hand limits (for details regarding the notation and the filtration we refer to [25]). Let $\mathcal{P}$ denote the set of equivalent martingale measures with respect to $P$. Since we assume the absence of arbitrage opportunities, it holds $\mathcal{P} \neq \emptyset$.

Recall that $\hat{Q}$ denotes the set of all probability measures on $(\Omega, \mathcal{F})$ absolutely continuous with respect to $P$. For $Q \in \hat{Q}$ we denote the expectation with respect to $Q$ by $E^Q$ and the Radon-Nikodym derivative $dQ/dP$ by $Z_Q$. Let us denote $Z_P := \{Z^*_P : P^* \in \mathcal{P}\}$.

A self-financing strategy is given by an initial capital $V_0 \geq 0$ and a predictable process $\xi$ such that the resulting value process $V_t = V_0 + \int_0^t \xi_s dS_s, \quad t \in [0,T],$ is well defined. Such a strategy $(V_0, \xi)$ is called admissible if the corresponding value process $V$ satisfies $V_t \geq 0$ for all $t \in [0,T]$.

Consider a contingent claim. Its payoff is given by an $\mathcal{F}_T$-measurable, nonnegative random variable $H \in L^1$. We assume

$$U_0 = \sup_{P^* \in \mathcal{P}} E^{P^*}[H] < +\infty. \quad (4.1)$$

The above equation is the dual characterization of the superhedging price $U_0$, the smallest amount $V_0$ such that there exists an admissible strategy $(V_0, \xi)$ with value process $V_t$ satisfying $V_T \geq H$ (see [13] for an overview over this topic). In the complete case, where the equivalent martingale measure $P^*$ is unique, $U_0 = E^{P^*}[H]$ is the unique arbitrage-free price of the contingent claim.

Since superhedging can be quite expensive in the incomplete market (see [21] for the general semimartingale case), we search for the best hedge an investor can achieve with a smaller amount $\tilde{V}_0 < U_0$. In other words, we look for an admissible strategy $(V_0, \xi)$ with $0 < V_0 \leq \tilde{V}_0$ that minimizes the risk of losses due to the shortfall $\{\omega : V_T(\omega) < H(\omega)\}$, this means we want to minimize the risk of $-(H - V_T)^+$. The risk will be measured by a suitable risk measure $\rho$. Thus, we consider the dynamic optimization problem of finding an admissible strategy that minimizes

$$\min_{(V_0, \xi)} \rho \left( - (H - V_T)^+ \right) \quad (4.2)$$
under the capital constraint of investing less capital than the superhedging price

\[ 0 < V_0 \leq \tilde{V}_0 < U_0. \]  

(4.3)

The dynamic optimization problem (4.2), (4.3) can be split into the following two problems:

1. Static optimization problem: Find an optimal modified claim \( \tilde{\varphi}H \), where \( \tilde{\varphi} \) is a randomized test solving

\[ \min_{\varphi \in R_0} \rho((\varphi - 1)H), \]  

(4.4)

\[ R_0 = \{ \varphi : \Omega \to [0,1], \mathcal{F}_T \text{ - measurable}, \sup_{P^* \in \mathcal{P}} E^{P^*}[\varphi H] \leq \tilde{V}_0 \}. \]  

(4.5)

2. Representation problem: Find a superhedging strategy for the modified claim \( \tilde{\varphi}H \).

The representation problem can be solved by the optional decomposition theorem of Föllmer and Kabanov [14] (see Appendix, Theorem C.3). The idea of splitting the dynamic optimization problem in this way was introduced by Föllmer and Leukert [16], minimizing the probability of a shortfall. It was used for the expectation of a loss function as risk measure in [17], for coherent risk measures in Nakano [31, 32], Rudloff [36] and for convex risk measures in Rudloff [38] analogously. The only property of \( \rho \) that is needed in the proof is monotonicity.

**Theorem 4.1.** Let \( \rho : L^1 \to \mathbb{R} \cup \{+\infty\} \) be a monotone function and let \( \tilde{\varphi} \) be a solution of the minimization problem (4.4) and \((\tilde{V}_0, \tilde{\xi})\) be the admissible strategy, where \( \tilde{\xi} \) is determined by the optional decomposition of the claim \( \tilde{\varphi}H \). Then the strategy \((\tilde{V}_0, \tilde{\xi})\) solves the optimization problem (4.2), (4.3) and it holds

\[ \min_{(V_0, \xi)} \rho(-(H - V_T)^+) = \min_{\varphi \in R_0} \rho((\varphi - 1)H). \]  

(4.6)

To prove the theorem, we first review the optional decomposition theorem (Theorem C.3) in our setting (see also [16], [17]). Therefore, we consider the modified claim \( \tilde{\varphi}H \), where \( \tilde{\varphi} \) is the solution of (4.4) and define \( \tilde{U} \) as a right-continuous version of the process

\[ \tilde{U}_t = \text{ess. sup}_{P^* \in \mathcal{P}} E^{P^*}[\tilde{\varphi}H|\mathcal{F}_t]. \]

For the definition of the essential supremum, see Section C. The process \( \tilde{U} \) is a \( \mathcal{P} \)-supermartingale, i.e., a supermartingale with respect to any equivalent martingale measure \( P^* \in \mathcal{P} \) (see [16], [17]). By the optional decomposition theorem (Theorem C.3) there exists an admissible strategy \((\tilde{U}_0, \tilde{\xi})\) and an increasing optional process \( \tilde{C} \) with \( \tilde{C}_0 = 0 \) such that

\[ \tilde{U}_t = \tilde{U}_0 + \int_0^t \tilde{\xi}_s dS_s - \tilde{C}_t. \]
One obtains that $\tilde{U}_0 = \sup_{P^* \in \mathcal{P}} E^{P^*}[\tilde{\varphi}H]$ is the superhedging price and $\tilde{\xi}$ the superhedging strategy of the modified claim $\tilde{\varphi}H$.

**Remark 4.2.** In the complete case where the equivalent martingale measure is unique, $(\tilde{U}_0, \tilde{\xi})$ is simply the replicating strategy for the modified claim $\tilde{\varphi}H$. Thus, $\tilde{U}_0 = E^{P^*}[\tilde{\varphi}H]$ is the unique arbitrage-free price of the contingent claim $\tilde{\varphi}H$.

**Remark 4.3.** In the incomplete market, when a risk measure $\rho$ is used that allows the construction of $\tilde{\varphi}$ via the Neyman-Pearson lemma directly (cf. [16] and some special cases of [17]), one can see that $\tilde{U}_0 = \tilde{V}_0$ since the optimal test $\tilde{\varphi}$ attains the bound $\tilde{V}_0$ in (4.5). In Theorem 4.9, equation (4.13) shows (except in the case where the dual solution takes only the value zero (see Remark 4.15 for the convex hedging case)) that in the general case the bound $\tilde{V}_0$ is as well attained by the optimal test.

**Proof of Theorem 4.1.** Let $(V_0, \xi)$ with $V_0 \leq \tilde{V}_0$ be an admissible strategy. We define the corresponding success ratio $\varphi = \varphi(V_0, \xi)$ as

$$\varphi(V_0, \xi) := 1_{\{V_T \geq H\}} + \frac{V_T}{H} 1_{\{V_T < H\}}.$$  

Thus, $-(H - V_T)^+ = (\varphi - 1)H$. Since $V_t$ is a $\mathcal{P}$-supermartingale and $\varphi H \leq V_T$:

$$\forall P^* \in \mathcal{P}: \quad E^{P^*}[\varphi H] \leq E^{P^*}[V_T] \leq V_0 \leq \tilde{V}_0,$$

hence, $\varphi \in R_0$. Thus,

$$\rho(-(H - V_T)^+) = \rho((\varphi - 1)H) \geq \rho((\tilde{\varphi} - 1)H), \quad (4.7)$$

where $\tilde{\varphi}$ is the solution to the static optimization problem (4.4). Inequality (4.7) is especially satisfied for the success ratio of the admissible strategy $(\nabla_0, \xi)$, where $\xi$ is the superhedging strategy for the modified claim $\tilde{\varphi}H$, determined by the optional decomposition theorem (Theorem C.3) and $\nabla_0 \in [\tilde{U}_0, V_0]$, where $\tilde{U}_0 = \sup_{P^* \in \mathcal{P}} E^{P^*}[\tilde{\varphi}H]$ is the superhedging price of the modified claim $\tilde{\varphi}H$. Thus,

$$\rho((\varphi_{(\nabla_0, \xi)} - 1)H) \geq \rho((\tilde{\varphi} - 1)H). \quad (4.8)$$

To show the reverse inequality, let us consider $\varphi_{(\nabla_0, \xi)}H = \min(\tilde{V}_T, H)$, where $\tilde{V}_T = \nabla_0 + \int_0^T \xi_s dS_s$. It holds

$$\tilde{V}_T = \nabla_0 + \int_0^T \xi_s dS_s = \nabla_0 + \tilde{U}_T + \tilde{C}_T - \tilde{U}_0 = \nabla_0 + \text{ess. sup}_{P^* \in \mathcal{P}} E^{P^*}[\tilde{\varphi}H|\mathcal{F}_T] + \tilde{C}_T - \tilde{U}_0 = \tilde{\varphi}H + \tilde{C}_T + \nabla_0 - \tilde{U}_0$$

$$\geq \tilde{\varphi}H.$$
4.1. HEDGING IN COMPLETE AND SPECIAL INCOMPLETE MARKETS

Thus, $\varphi(\tilde{V}_0, \tilde{\xi}) H \geq \tilde{\varphi} H$. Since $\rho$ is monotone, we obtain

$$\rho((\varphi(\tilde{V}_0, \tilde{\xi}) - 1)H) \leq \rho((\tilde{\varphi} - 1)H).$$

Together with (4.8), we see that $\varphi(\tilde{V}_0, \tilde{\xi})$ attains the minimum of the static optimization problem (4.4). Due to (4.7), we now have

$$\min_{(V_0, \xi)} \rho(-(H - V_T)^+) \geq \rho(-(H - \tilde{V}_T)^+).$$

Hence, $(\tilde{V}_0, \tilde{\xi})$ with $V_0 \in [\tilde{U}_0, \tilde{V}_0]$ is the strategy that attains the minimum in the dynamic optimization problem (4.2), (4.3) and it holds

$$\min_{(V_0, \xi)} \rho(-(H - V_T)^+) = \min_{\varphi \in \mathbb{R}_0} \rho((\varphi - 1)H).$$

Remark 4.4. As mentioned in Remark 4.3, relationship (4.13) of Theorem 4.9 means that $\tilde{U}_0 = \tilde{V}_0$. Thus, the optimal strategy is $(\tilde{V}_0, \tilde{\xi})$.

4.1 Hedging in Complete and Special Incomplete Markets

In this section, we consider the problem of hedging in complete markets, i.e., the set $\mathcal{P} = \{P^*\}$ is a singleton, when the investor is unwilling or unable to pay the unique
arbitrage free price of a contingent claim and wants to invest a sum less than this price. Since the results also hold true in the more general case of an incomplete market with \( Z_P := \{ Z^* : P^* \in P \} \) compact, we work in this more general setting.

We want to study the hedging problem using a risk measure as general as possible. In Theorem 4.1 we needed to assume that \( \rho \) is monotone. To solve the problem by application of the results of Chapter 2, we additionally need to assume that \( \rho \) is convex, lower semicontinuous and continuous in some \((\varphi_0 - 1)H \) with \( \varphi_0 \in R_0 \). First, we shall give the result for this most general case. Then, in the next subsections we shall add more properties to \( \rho \) which leads to different types of risk measures and we analyze the influence on the results.

4.1.1 The General Case

First, we consider a risk function as general as possible. We impose the following assumption.

**Assumption 4.5.** Let \( \rho : L^1 \rightarrow \mathbb{R} \cup \{+\infty\} \) be a monotone, convex, lower semicontinuous function, that is continuous and finite in some \((\varphi_0 - 1)H \) with \( \varphi_0 \in R_0 \) and satisfies \( \rho(0) < +\infty \).

**Remark 4.6.** Especially, if \( \rho \) is a lower semicontinuous convex function with \( \rho(Y) < +\infty \) for all \( Y \in L^1 \), then \( \rho \) is continuous for all \( Y \in L^1 \) since \( L^1 \) is a Banach space ([11], Corollary I.2.5).

In general, a lower semicontinuous convex function \( \rho : L^1 \rightarrow \mathbb{R} \cup \{+\infty\} \) is continuous in some \( Y \) if \( Y \) is an interior point of the domain of \( \rho \) (see [11], Corollary I.2.5).

Let us consider the measurable space \((Z_P, \mathcal{B})\), where \( \mathcal{B} \) is the \( \sigma \)-algebra of all Borel sets on \( Z_P \). We denote by \( \Lambda_+ \) the set of finite measures on \((Z_P, \mathcal{B})\).

**Remark 4.7.** We review the procedure deduced in Chapter 2 to solve the static optimization problem (4.4), where \( \rho \) is a function satisfying Assumption 4.5:

(i) Prove the existence of a solution \( \tilde{\varphi} \) to the primal problem (4.4) (Theorem 2.5)

\[
p = \min_{\varphi \in R_0} \rho((\varphi - 1)H) = \min_{\varphi \in R_0} \sup_{Y^* \in L^\infty_+} \{E[(1 - \varphi)HY^*] - \rho^*(-Y^*)]\}.
\]

(ii) Deduce the dual problem to (4.4) by Fenchel duality:

\[
d = \sup_{Y^* \in L^\infty_+} \{ \inf_{\varphi \in R_0} \{E[(1 - \varphi)HY^*] - \rho^*(-Y^*)\}\} \quad (4.9)
\]

and prove the validity of strong duality \( p = d \) (Theorem 2.6). We obtain the existence of a dual solution and can show that the problem is a saddle point problem.
4.1. HEDGING IN COMPLETE AND SPECIAL INCOMPLETE MARKETS

(iii) Consider the inner problem of the dual problem (4.9) for an arbitrary $Y^* \in L^\infty$: 

$$p^i(Y^*) := \max_{\varphi \in \mathbb{K}_0} E[\varphi HY^*].$$  \hspace{1cm} (4.10)

Prove the existence of a solution $\tilde{\varphi}_{Y^*}$ to (4.10) (Lemma 2.7). Deduce the dual problem by Fenchel duality:

$$d^i(Y^*) = \inf_{\lambda \in \Lambda_+} \left\{ \int_\Omega [HY^* - H \int_\mathcal{P} Z_{P^*} d\lambda] + dP + \tilde{V}_0 \lambda(Z_{P^*}) \right\}. \hspace{1cm}$$

Prove the validity of strong duality $p^i(Y^*) = d^i(Y^*)$ and deduce the necessary and sufficient structure of a solution $\tilde{\varphi}_{Y^*}$ to the inner problem (4.10) (Theorem 2.8). Note that $p^i(Y^*) = -\tilde{p}(-Y^*)$ in the notation of Chapter 2.

(iv) Apply Theorem 2.6 and 2.8 to the primal problem (4.4) and deduce the necessary and sufficient structure of a solution $\tilde{\varphi}$ to (4.4) (Theorem 2.9).

Remark 4.8. Problem (4.10) can be identified as a problem of test theory. Let us define the measures $O$ and $O^* = O^*(P^*)$ by $\frac{dO}{dP} = HY^*$ and $\frac{dO^*}{dP^*} = H$ for $P^* \in \mathcal{P}$. Problem (4.10) turns into

$$\max_{\varphi \in \mathbb{R}} E^O[\varphi] \text{ subject to } \forall P^* \in \mathcal{P} : E^{O^*}[\varphi] \leq \tilde{V}_0 =: \alpha.$$

This is equivalent of looking for an optimal test $\tilde{F}_{Y^*}$ when testing the compound hypothesis $H_0 = \{O^*(P^*) : P^* \in \mathcal{P}\}$, parameterized by the class of equivalent martingale measures, against the simple alternative $H_1 = \{O\}$ in a generalized sense. In the generalized test problem (see Section 3.2), $O$ and $O^*$ are not necessarily probability measures, but measures and the significance level $\alpha$ is generalized to be a positive continuous function $\alpha(P^*)$.

We now give the main results by applying the procedure described in Remark 4.7 to problem (4.4).

Theorem 4.9 (Solution to the Generalized Hedging Problem). Let $\rho$ be as in Assumption 4.5 and let $Z_{\mathcal{P}}$ be compact. Then, there exists a solution $\tilde{\varphi}$ to (4.4). If $\rho$ is strictly convex, then any two solutions coincide $P - a.s.$ on $\{H > 0\}$. There exists a pair $(\bar{Y}^*, \bar{\lambda})$ solving

$$\max_{Y^* \in L^\infty_+ : \lambda \in \Lambda_+} \left\{ E[HY^* \wedge H \int_\mathcal{P} Z_{P^*} d\lambda] - \tilde{V}_0 \lambda(Z_{P^*}) - \rho^*(-Y^*) \right\}, \hspace{1cm} (4.11)$$

where $x \wedge y = \min(x, y)$. It follows that:
The solution of the static optimization problem (4.4) is
\[ \tilde{\varphi} = \begin{cases} 1 : & H\tilde{Y} > H \int_P Z_P d\tilde{\lambda} \quad P - a.s. \quad (4.12) \\ 0 : & H\tilde{Y} < H \int_P Z_P d\tilde{\lambda} \end{cases} \]
with
\[ E[H^\varphi] = \tilde{V}_0 \quad \tilde{\lambda} - a.s. \quad (4.13) \]

\[ (\tilde{\varphi}, \tilde{Y}^*) \text{ is a saddle point of the functional } (\varphi, Y^*) \mapsto E[(1 - \varphi)HY^*] - \rho^*(-Y^*) \]
in \( R_0 \times L_1^\infty \).

\[ (\tilde{V}_0, \tilde{\xi}) \text{ solves the dynamic hedging problem (4.2), (4.3), where } \tilde{\xi} \text{ is the superhedging strategy of the modified claim } \tilde{\varphi} H, \text{ obtained by the optional decomposition theorem (Theorem C.3)}. \]

Remark 4.10. It follows that there exists a \([0, 1]\)-valued random variable \( \delta \) such that \( \tilde{\varphi} \) as in Theorem 4.9 satisfies
\[ \tilde{\varphi}(\omega) = 1_{\{H\tilde{Y} > H \int_P Z_P d\tilde{\lambda}\}}(\omega) + \delta(\omega)1_{\{H\tilde{Y} = H \int_P Z_P d\tilde{\lambda}\}}(\omega). \]
\( \delta \) has to be chosen such that \( \tilde{\varphi} \) satisfies (4.13).

Remark 4.11. Theorem 4.9 gives a result about the structure of a solution to the hedging problem for every risk measure satisfying Assumption 4.5. Note that we do not need a translation property for \( \rho \) to obtain this result.

Proof of Theorem 4.9
1) We can apply the theory of Chapter 2 by setting \( H \) equal to the payoff of the contingent claim in \( L_1^\infty \), \( A\varphi = H\varphi \), \( b = -H \), \( C^* = Z_P \) and \( c = \tilde{V}_0 \). We have \( X = L_1^\infty \), endowed with the norm topology, \( X^* = ba(\Omega, \mathcal{F}, P) \), \( Y = L_1^\infty \), \( Y^* = L_1^\infty \). The function \( \rho \) is as in Assumption 4.5 and \( X_1 \) is the set of randomized tests and coincides with \( R = \{ \varphi : \Omega \to [0, 1], \mathcal{F}_T \text{ - measurable}\} \). Hence, \( X_0 = R_0 \). Then, the static optimization problem (4.4) can be identified as a special case of problem (2.1).

2) First, we verify that condition (A1)-(A7) of Assumption 2.1 are satisfied:

(A1): \( c = \tilde{V}_0 > 0 \) (see (4.3)).

(A2): \( H \in L_1^\infty, C^* = Z_P \subseteq L_1^\infty \) and \( \{HX^* : X^* \in C^*\} = \{HZ_P^* : P^* \in P\} \subseteq L_1^\infty \), since we assumed in (4.1) the superhedging price of \( H \) to be finite.

(A3): (4.1) also ensures that \( \sup_{X^* \in C^*} \|HX^*\|_{L_1} < +\infty \). The set \( Z_P \) is assumed to be compact.

(A4): The operator \( A : L_1^\infty \to L_1^\infty \), defined by \( A\varphi := H\varphi \), is linear and continuous.
Since the existence of a solution we prove that the function $f : L^\infty \to \mathbb{R} \cup \{+\infty\}$, defined by $f(\varphi) := \rho((\varphi - 1)H)$, is lower semicontinuous in the weak* topology. Because of Assumption 4.5, $\rho$ admits a dual representation (see Theorem 1.5 (b))

$$\rho(Y) = \sup_{Y^* \in L^\infty} \{E[-YY^*] - \rho^*(-Y^*)\}.$$ 

Thus,

$$f(\varphi) = \sup_{Y^* \in L^\infty} \{E[(1 - \varphi)HY^*] - \rho^*(-Y^*)\} = \sup_{Y^* \in L^\infty} \{E[HY^*] + E[\varphi H(-Y^*)] - \rho^*(-Y^*)\}.$$ 

The function $\varphi \mapsto E[\varphi H(-Y^*)] + E[HY^*] - \rho^*(-Y^*)$ is weakly* continuous for all $Y^* \in L^\infty$ since $H(-Y^*) \in L^1$. Since $f(\varphi)$ is the pointwise supremum of weakly* continuous functions, $f$ is weakly* lower semicontinuous (Lemma 2.38, [2]).

(A7): The map $\varphi \mapsto \langle Y^*, A\varphi \rangle = E[Y^* H\varphi]$ is continuous in the weak* topology for all $Y^* \in L^\infty$, since $HY^* \in L^1$ for all $Y^* \in L^\infty$.

Thus, all conditions in Assumption 2.1 are satisfied.

3) The existence of a solution $\tilde{x}$ to (4.4) follows from Theorem 2.5. If $\rho$ is additionally strictly convex, then any two solutions coincides $P$-a.s. on $\{H > 0\}$. By definition of the adjoint operator $A^*$ of $A$ (see Definition 6.51 in [2]), the equation $\langle A\varphi, Y^* \rangle = \langle \varphi, A^*Y^* \rangle$ has to be satisfied for all $\varphi \in L^\infty, Y^* \in L^\infty$.

Since from the validity of (A7) we obtain $A^*Y^* \in L^1$ for all $Y^* \in L^\infty$ (cf. Remark 2.2), it holds

$$\forall \varphi \in L^\infty, \forall Y^* \in L^\infty : \int_\Omega H\varphi Y^* dP = \int_\Omega \varphi A^*Y^* dP. \quad (4.14)$$

Suppose $A^*Y^* < HY^*$ on $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) > 0$. Define $\varphi(\omega) = 1_{\Omega_1}(\omega)$. This $\varphi \in L^\infty$ violates (4.14). The case $A^*Y^* > HY^*$ on $\Omega_2 \subseteq \Omega$ with $P(\Omega_2) > 0$ is analogous. We conclude $A^*Y^* = HY^* = AY^*$, i.e., the operator $A$ is self-adjointed.

In our setting, the optimization problem (2.23) becomes (4.11). Note that, since $\rho$ is monotone, it is convenient to work with $-Y^* \in \text{dom} \rho^*$ (cf. Theorem 1.5), whereas in Chapter 2 we work with $Y^* \in \text{dom} \rho^*$. By applying Theorem 2.9, we obtain the existence of an optimal pair $(\tilde{Y}^*, \lambda)$ solving (4.11) and the structure (4.12), (4.13) of an optimal randomized test $\tilde{\varphi}$. Furthermore, $(\tilde{\varphi}, \tilde{Y}^*)$ is a saddle point of the functional $(\varphi, Y^*) \mapsto E[(1 - \varphi)HY^*] - \rho^*(-Y^*)$. 

4.1. HEDGING IN COMPLETE AND SPECIAL INCOMPLETE MARKETS
4) Equation (4.13) and Theorem 4.1 show, that \((\tilde{V}_0, \tilde{\xi})\) solves the dynamic hedging problem (4.2), (4.3), where \(\tilde{\xi}\) is the superhedging strategy of the modified claim \(\tilde{\varphi}H\) obtained by the optional decomposition theorem (Theorem C.3).

If \(\rho\) satisfies additionally to Assumption 4.5 the translation property and \(\rho(0) = 0\), it forms a convex risk measure (see Section 1.2). If it is additionally to this positively homogeneous, it is a coherent risk measure (Section 1.3). The translation property is a natural assumption for a risk measure used as a risk adjusted capital requirement, but is not necessary for the proof of Theorem 4.9. Furthermore, there are risk measures, that do not necessarily have this property, e.g. the expectation of a loss function. This risk measure was used in the context of hedging in [17].

In the following subsections, we shall analyze the hedging problem (4.2) and therefore the corresponding static optimization problem (4.4) using different important risk measures to quantify the shortfall risk. In Section 4.1.2, we use convex risk measures and in Section 4.1.3 coherent risk measures. These risk measures will be special cases of functions satisfying Assumption 4.5. We shall analyze the influence of different additional properties of these risk measures on the results of Theorem 4.9. Furthermore, we shall compare these results with results that can be found in the recent literature using these special risk measures when hedging in incomplete markets. We shall show that Theorem 4.9 is widely applicable and that the obtained results improve previous results in the case \(Z_P\) compact.

In Section 4.1.4, we shall consider the hedging problem when the risk is measured by a robust version of the expectation of a loss function. For Lipschitz continuous loss functions the problem can be solved by an application of Theorem 4.9. We show that the linear case is related to the coherent hedging problem and can be solved analogously. We compare our results with the literature. The case of a general loss function turns out to fit not exactly to the setting of Theorem 4.9. We show which assumptions can be weakened and give proposals how the problem could be solved in general.

We start with the case of convex risk measures.

4.1.2 Convex Hedging

In this section we consider the problem of hedging when the attitude towards losses is modelled by a convex risk measure. This problem was studied in Rudloff [37, 38].

**Assumption 4.12.** Let \(\rho : L^1 \to \mathbb{R} \cup \{+\infty\}\) be a lower semicontinuous convex risk measure that is continuous and finite in some \((\varphi_0 - 1)H\) with \(\varphi_0 \in \mathbb{R}_0\).

**Remark 4.13.** Note that, if \(\rho(Y) < +\infty\) for all \(Y \in L^1\), a lower semicontinuous convex risk measure turns out to be continuous (see Remark 4.6). Finite valued convex risk measures are discussed in [18], [19], where also examples can be found.
A convex risk measure \( \rho \) is lower semicontinuous if and only if its acceptance set \( A_\rho \) is closed (Proposition 1.8 (vi)).

Convex risk measures have been studied in Section 1.2 and are by definition convex, monotone, satisfy the translation property and \( \rho(0) = 0 \). Lower semicontinuous convex risk measures on \( L^1 \) admit the following dual representation (Theorem 1.16)

\[
\rho(Y) = \sup_{Q \in \mathcal{Q}} \left\{ \mathbb{E}^Q[-Y] - \sup_{Y \in A_\rho} \mathbb{E}^Q[-\tilde{Y}] \right\},
\]

where \( \mathcal{Q} := \{ Q \in \hat{\mathcal{Q}} : Z_Q \in L^\infty \} \) is the set of all probability measures \( Q \), absolutely continuous to \( P \) and with densities in \( L^\infty \) and \( A_\rho \) is the acceptance set of \( \rho \).

The dynamic convex hedging problem consists in finding an admissible strategy solving

\[
\min_{(V_0, \xi)} \left\{ - \rho(H - V_T) \right\}, \quad 0 < V_0 \leq \tilde{V}_0 < U_0.
\]

With Theorem 4.1, it follows that the corresponding static optimization problem is

\[
\min_{\varphi \in R_0} \left\{ \rho((\varphi - 1)H) = \sup_{\varphi \in R_0} \left\{ \mathbb{E}^{Q\varphi}( (1 - \varphi)H ) - \sup_{Y \in A_\rho} \mathbb{E}^Q( -Y ) \right\} \right\},
\]

where \( R_0 \) is as in (4.5). By applying Theorem 4.9 and Theorem 1.16, we obtain the following result:

**Corollary 4.14 (Convex Hedging).** Let \( \rho \) be a convex risk measure satisfying Assumption 4.12 and let \( Z_P \) be compact. Then, there exists a solution \( \tilde{\varphi} \) to (4.17). Furthermore, there exists a pair \((\tilde{Q}, \tilde{\lambda}) \in Q \times \Lambda_+\) solving

\[
\max_{Q \in \mathcal{Q}, \lambda \in \Lambda_+} \left\{ \mathbb{E}[H Z_Q \wedge H \int_P Z_P \lambda ] - \tilde{V}_0 \lambda(Z_P) - \sup_{Y \in A_\rho} \mathbb{E}^Q( -Y ) \right\}.
\]

It follows that:

- The solution of the static optimization problem (4.17) is

\[
\tilde{\varphi} = \begin{cases} 
1 : & H \tilde{Z}_Q > H \int_P Z_P \lambda \\
0 : & H \tilde{Z}_Q < H \int_P Z_P \lambda
\end{cases} \quad P \text{- a.s.}
\]

with

\[
E^{P_\varphi}[\tilde{\varphi}H] = \tilde{V}_0 \quad \tilde{\lambda} \text{- a.s.}
\]

- \((\tilde{\varphi}, \tilde{Q})\) is a saddle point of the functional \((\varphi, Q) \mapsto \mathbb{E}^{Q\varphi}( (1 - \varphi)H ) - \sup_{Y \in A_\rho} \mathbb{E}^Q( -Y )\) in \( R_0 \times \mathcal{Q} \).

- \((\tilde{V}_0, \tilde{\xi})\) solves the dynamic convex hedging problem (4.16), where \( \tilde{\xi} \) is the superhedging strategy of the modified claim \( \tilde{\varphi}H \), obtained by the optional decomposition theorem (Theorem C.3).
It is not longer possible to show the essential uniqueness of a solution $\tilde{\varphi}$ to (4.17) on $\{H > 0\}$ since a convex risk measure cannot be strictly convex. The translation property of $\rho$ and $\rho(0) = 0$ imply the linearity of $\rho$ on the one dimensional subspace $L(1)$ spanned by the random variable $1$ (see Proposition 1.14). This means that for convex risk measures one can only show the existence, but not the essential uniqueness of a solution.

Proof of Corollary 4.14. Since $\rho$ satisfies Assumption 4.12, we can apply Theorem 4.9. Together with the dual representation of convex risk measures (see Theorem 1.16) we obtain the stated results. 

Remark 4.15. From equation (4.19) it follows (except in the case where $\tilde{\lambda}$ is the zero-measure, i.e., $\tilde{\lambda}(B) = 0$ for all $B \in \mathcal{B}$) that $\tilde{U}_0 := \sup_{P^* \in P} E^{P^*}[\tilde{\varphi}H] = \tilde{V}_0$ (see Remark 4.3 and 4.4). This ensures, that $\tilde{V}_0$ is the minimal amount of capital that is necessary to solve together with $\tilde{\xi}$ the dynamic problem (4.16).

Let us check if the case $\tilde{\lambda}(B) = 0$ for all $B \in \mathcal{B}$ can be excluded. If $\tilde{\lambda}$ is the zero-measure, the optimal randomized test is

$$\tilde{\varphi} = \begin{cases} 1 & : H\tilde{Z}_Q > 0 \\ \delta & : H\tilde{Z}_Q = 0 \end{cases} \quad P - \text{a.s.},$$

where $\delta$ is a $[0, 1]$-valued random variable such that (if possible) $\tilde{\varphi} \in R_0$, for instance $\delta = 0$. Equation (4.19) has no longer an impact on $\tilde{\varphi}$ since $\tilde{\lambda}$ takes only the value zero. Then, the optimal value of the static optimization problem (4.17) becomes (see Theorem 1.5 (a), (c))

$$p = \rho(\tilde{\varphi}H - H) = E^{\tilde{Q}}[H] - E^{\tilde{Q}}[\tilde{\varphi}H] - \rho^*(-\tilde{Z}_Q) = -\rho^*(-\tilde{Z}_Q) \leq 0.$$ 

From $\rho$ monotone and $\tilde{\varphi}H - H \leq 0$, we obtain $\rho(\tilde{\varphi}H - H) \geq 0$ and thus $\rho(\tilde{\varphi}H - H) = 0$. This means, the risk of the difference between the modified claim $\tilde{\varphi}H$ and $H$ is zero. In some special cases we can exclude that $\tilde{\lambda}$ takes only value zero. If $\tilde{Q}$ is a probability measure equivalent to $P$, then $\tilde{\lambda}(B) = 0$ for all $B \in \mathcal{B}$ implies $\tilde{\varphi} \notin R_0$. Thus, in this case $\tilde{\lambda}(B) = 0$ for all $B \in \mathcal{B}$ is not possible.

4.1.3 Coherent Hedging

In this section, we consider the hedging problem when the risk of losses due to the shortfall is measured by a coherent risk measure. This problem was studied in Nakano [31, 32] and Rudloff [36]. Coherent risk measures are convex risk measures that are additionally positively homogeneous. In this section, we deduce the main results for the case $Z_P$ compact and show the differences between the method used in [31, 32] and our method to solve the problem. We show that our results give more information about the structure of a solution. A comparison of the results in the general incomplete market can be found in Section 4.2.2.
4.1. HEDGING IN COMPLETE AND SPECIAL INCOMPLETE MARKETS

Assumption 4.16. Let $\rho : L^1 \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous coherent risk measure that is continuous and finite in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$.

To assume that $\rho$ is continuous in some $(\varphi_0 - 1)H$ with $\varphi_0 \in R_0$ is not very restrictive. If we take for instance a finite valued lower semicontinuous coherent risk measure $\rho$ as considered for example in [19], then $\rho$ is continuous. (see Remark 4.6).

The dynamic coherent hedging problem is to find an admissible strategy solving

$$\min_{(V_0, \xi)} \rho \left( - (H - V_T)^+ \right), \quad 0 < V_0 \leq \tilde{V}_0 < U_0. \quad (4.20)$$

With Theorem 4.1 and the dual representation of a lower semicontinuous coherent risk measure (Theorem 1.25) it follows that the corresponding static optimization problem, the primal problem, is

$$\min_{\varphi \in R_0} \rho ((\varphi - 1)H) = \min \{\sup_{Q \in \mathcal{Q}} E^Q [(1 - \varphi)H]\}, \quad (4.21)$$

where

$$R_0 = \{\varphi : \Omega \to [0, 1], \mathcal{F}_T - \text{measurable}, \sup_{P^* \in P} E^{P^*}[\varphi H] \leq \tilde{V}_0\} \quad (4.22)$$

and $\mathcal{Q}$, the maximal representing set, is a convex and weakly* closed subset of $\{Q \in \hat{Q} : Z_Q \in L^\infty\}$ determined by the dual representation of $\rho$ (Theorem 1.25).

The dual problem of (4.21) is (see Remark 4.7, (ii))

$$d = \max_{Q \in \mathcal{Q}, \varphi \in R_0} \{\min_{\varphi \in R_0} E^Q [(1 - \varphi)H]\}. \quad (4.23)$$

The inner problem of (4.23) for a fixed $Q \in \mathcal{Q}$ is (see Remark 4.7, (iii))

$$p^i(Q) := \max_{\varphi \in R_0} E^Q [\varphi H]. \quad (4.24)$$

Its dual problem (see Remark 4.7, (iii)), deduced via Fenchel duality, is

$$d^i(Q) = \inf_{\lambda \in \Lambda_+} \left\{ \int_{\Omega} [HZ_Q - H \int_{P} Z_{P^*} d\lambda]^+ dP + \tilde{V}_0 \lambda (Z_P) \right\}. \quad (4.25)$$

Corollary 4.17 (Coherent Hedging). Let $\rho$ be a coherent risk measure satisfying Assumption 4.16 and let $Z_P$ be compact. Then, there exists a solution $\bar{\varphi}$ to (4.21). Furthermore, there exists a pair $(\bar{Q}, \bar{\lambda}) \in \mathcal{Q} \times \Lambda_+$ solving

$$\max_{Q \in \mathcal{Q}, \lambda \in \Lambda_+} \{E[HZ_Q \wedge H \int_{P} Z_{P^*} d\lambda] - \tilde{V}_0 \lambda (Z_P)\}. \quad (4.26)$$

It follows that:
• The solution of the static optimization problem (4.21) is

\[ \tilde{\varphi} = \begin{cases} 
1 & : \ H \tilde{Z}_Q > H \int P Z_P d\tilde{\lambda} \\
0 & : \ H \tilde{Z}_Q < H \int P Z_P d\tilde{\lambda} 
\end{cases} \quad P \text{- a.s.} \tag{4.27} \]

with

\[ E^{\tilde{P}}[\tilde{\varphi} H] = \tilde{V}_0 \quad \tilde{\lambda} \text{- a.s.} \tag{4.28} \]

• \((\tilde{\varphi}, \tilde{Q})\) is a saddle point of the functional \((\varphi, Q) \mapsto E^Q[(1 - \varphi)H]\) in \(R_0 \times Q\).

• \((\tilde{V}_0, \tilde{\xi})\) solves the dynamic coherent hedging problem (4.20), where \(\tilde{\xi}\) is the superhedging strategy of the modified claim \(\tilde{\varphi} H\), obtained by the optional decomposition theorem (Theorem C.3).

Proof. Since coherent risk measures are also convex risk measures, the results follow from Corollary 4.14 and the dual representation of coherent risk measures (see Theorem 1.25).

The problem (4.20) of hedging with coherent risk measures was studied by Nakano [31, 32]. In [32], the decomposition of the dynamic problem and the existence of a solution to the static problem was shown. In [31] a similar result as in Corollary 4.17 was obtained. We now want to make the differences in the methods clear that are used in the proofs and show in which way Corollary 4.17 is an improvement of Theorem 4.11 in [31] for the case \(Z_P\) compact. Nakano [31] followed the method of Cvitanić and Karatzas [7] (see Section 3.1) to show that the solution of the static optimization problem is a Neyman-Pearson test. In Nakano [31] it is necessary to introduce the enlarged sets

\[ Z = \{Z \in L^\infty_+ | E[Z] \leq 1, \forall X \in L^1_+ : E[XZ] \leq \rho(-X)\} \supseteq \{Z_Q : Q \in Q\} \]

and

\[ D = \{D \in L^1_+ | E[D] \leq 1, E[D H] \leq U_0, \forall \varphi \in R_0 : E[D \varphi H] \leq \tilde{V}_0\} \supseteq Z_P, \]

where \(Z\) is closed under \(P - \text{a.s.}\) convergence and convex and \(D\) is bounded in \(L^1\), convex and closed under \(P - \text{a.s.}\) convergence. These enlarged sets were introduced to ensure the existence of a quadruple \((\tilde{Z}, \tilde{D}, \tilde{z}, \tilde{\varphi}) \in (Z \times D \times (0, \infty) \times R_0)\) that yield equality in

\[ \forall Z \in Z, \forall D \in D, \forall z > 0, \forall \varphi \in R_0 : \ E[Z(H - \varphi H)] \geq E[H(Z \wedge z D)] - \tilde{V}_0 z \tag{4.29} \]

In Theorem 4.11 in [31] the typical 0-1-structure of an optimal randomized test \(\tilde{\varphi}\) is deduced, but with respect to elements from the larger sets \(Z\) and \(D:\)

\[ \tilde{\varphi} = 1_{\{zD < Z\}} + \delta 1_{\{zD = Z\}}, \]
4.1. HEDGING IN COMPLETE AND SPECIAL INCOMPLETE MARKETS

where \( (\hat{z}, \hat{Z}, \hat{D}) \) attain the supremum of

\[
\sup_{z \geq 0, Z \in \mathcal{Z}, D \in \mathcal{D}} \{ E[H(Z \wedge zD)] - \tilde{V}_0 z \}.
\]

We can show that inequality (4.29) corresponds to the validity of weak duality between the inner problem (4.24) and its dual problem (4.25) that is automatically satisfied (cf. Theorem A.12). That is

\[
\forall Q \in \mathcal{Q}, \forall \lambda \in \Lambda_+, \forall \varphi \in R_0 : \quad E^Q[(1 - \varphi)H] \geq E[H_{ZQ} \wedge H \int_{\mathcal{P}} Z_P d\lambda] - \tilde{V}_0 \lambda(Z_P).
\]

With our method, it is not necessary to consider the larger sets \( \mathcal{Z} \) and \( \mathcal{D} \). We prove the validity of strong duality via Fenchel duality directly (step (ii) and (iii), Remark 4.7). The existence of a dual solution follows from the validity of strong duality (cf. Theorem A.12). This makes it possible to deduce the 0-1-structure of \( \tilde{\varphi} \) with respect to elements from the original sets \( \mathcal{Q} \) and \( \mathcal{P} \). In contrast to this, Nakano [31] proved the existence of a solution to the dual problem. Therefore, it was necessary to consider the larger sets \( \mathcal{Z} \) and \( \mathcal{D} \). The application of Corollary 4.17 shows that there is a one-to-one relationship between the optimal elements \( \hat{Z}, \hat{D} \) and \( \hat{z} \) of [31] and elements of \( \mathcal{Q} \) and \( \mathcal{P} \):

\[
\hat{Z} = \begin{cases} 
\tilde{Z}_Q & : \{ H > 0 \} \\
0 & : \{ H = 0 \}
\end{cases},
\]

\[
\hat{D} = \begin{cases} 
k \int_{\mathcal{P}} Z_P d\tilde{\lambda} & : \{ H > 0 \} \\
0 & : \{ H = 0 \}
\end{cases},
\]

\[
\hat{z} = \tilde{\lambda}(Z_P),
\]

where \( (\tilde{Q}, \tilde{\lambda}) \) is the optimal pair in (4.26) and \( k = \tilde{\lambda}(Z_P)^{-1} \) if \( \tilde{\lambda}(Z_P) \neq 0 \) and zero if \( \tilde{\lambda}(Z_P) = 0 \). It holds \( \tilde{\varphi} = \tilde{\varphi} \). Thus, the direct application of convex duality gives more detailed information about the structure of the optimal randomized test \( \tilde{\varphi} \). Another difference to [31] is that we consider coherent risk measures that can also attain the value +\( \infty \). Furthermore, we now can show in equation (4.28) of Corollary 4.17 that the upper bound of the constraint in (4.22) is attained (except in the pathological case where \( \tilde{\lambda} \) takes only the value zero (see Remark 4.15)). Then, \( \tilde{U}_0 = \sup_{P \in \mathcal{P}} E^P[\tilde{\varphi}H] = \tilde{V}_0 \). It follows, that \( \tilde{V}_0 \), the upper capital boundary, is the minimal required capital that is necessary for the optimal hedge and thus, \( (\tilde{V}_0, \tilde{\xi}) \) solves the optimization problem (4.20) (see Proof of Theorem 4.1 and Remark 4.4). This was not possible to deduce from the analogous result \( E[\tilde{\varphi}H \hat{D}] = \tilde{V}_0, \hat{D} \in \mathcal{D} \) in Nakano [31]. A comparison of the results in the general incomplete market can be found in Section 4.2.2.
4.1.4 Robust Efficient Hedging

In the concept of efficient hedging the expectation of a loss function $l$ is used as the risk measure in problem (4.2). This problem was introduced by Föllmer and Leukert [17] (see also [19]).

**Assumption 4.18.** Let $l : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing convex function with $l(x) = 0$ for all $x \leq 0$.

The function $l$ is continuous since it is a convex and finite valued function on $\mathbb{R}$ ([11], Corollary I.2.3). Let $L^0 = L^0(\Omega, \mathcal{F}, P)$ be the space of $P$-a.s. finite random variables and $L^0_+ := \{Y \in L^0 : Y \geq 0 \text{ } P\text{-a.s.}\}$. We define $L : L^1 \rightarrow L^0_+$ by

$$L(Y)(\omega) := l(Y(\omega)).$$

The function $L$ maps into $L^0_+$ since $l$ is continuous and maps into $\mathbb{R}_+$. We consider the dynamic efficient hedging problem derived from (4.2) with the risk measure $ho_0(Y) = E[L(\cdot)]$ for $Y \in L^1$. This means, we look for an admissible strategy that is a solution of

$$\min_{(V_0, \xi)} E[L((H - V_T)^+)], \quad 0 < V_0 \leq \tilde{V}_0 < U_0.$$  

We want to generalize this problem and consider a robust version (see Remark 8.12 in [19]) of the expectation of a loss function defined as

$$\rho_1(Y) = \sup_{Q \in \mathcal{Q}} E^Q[L(\cdot)], \quad Y \in L^1,$$  \hspace{1cm} (4.30)

where $\mathcal{Q} \subseteq \hat{\mathcal{Q}}$ is a set of probability measures absolutely continuous with respect to $P$. By passing from a single probability measure $P$ to a whole set $\mathcal{Q}$ of probability measures one can take into account an uncertainty regarding the underlying model. This can be the case if for instance the underlying asset price process is modelled via an jump-diffusion process and there is uncertainty regarding the jump intensities (see [26] for several examples).

In the following, we will study the robust efficient hedging problem using the risk measure $\rho_1$. The dynamic problem is to find an admissible strategy solving

$$\min_{(V_0, \xi)} \sup_{Q \in \mathcal{Q}} E^Q[L((H - V_T)^+)], \quad 0 < V_0 \leq \tilde{V}_0 < U_0.$$  \hspace{1cm} (4.31)

**Remark 4.19.** A special case of (4.31) is the problem of quantile hedging. In this case, the probability of losses due to the shortfall has to be minimized. We obtain this problem from (4.31) by setting $\mathcal{Q} = \{P\}$ and using the non-convex loss function $l(x) = 1_{(0,\infty)}(x)$. This problem was solved in [16].

We impose the following assumption on $l$, $\mathcal{Q}$ and $H$. 

4.1. HEDGING IN COMPLETE AND SPECIAL INCOMPLETE MARKETS

**Assumption 4.20.** \( \sup_{Q \in \mathcal{Q}} E^Q[L(H)] < +\infty. \)

Let us analyze the properties of \( \rho_1 \).

**Proposition 4.21.** Under Assumption 4.18, the function \( \rho_1 : L^1 \to \mathbb{R} \cup \{+\infty\} \) is monotone, convex, lower semicontinuous and satisfies \( \rho_1(0) = 0 \).

**Proof.** \( \rho_1 \) is monotone and convex since \( l \) is nondecreasing and convex. \( \rho_1 \) satisfies \( \rho_1(0) = 0 \) since \( l(0) = 0 \). To prove the lower semicontinuity of \( \rho_1 \), we prove that \( \text{epi } \rho_1 \) is closed. Take a sequence \((Y_n, r_n) \in \text{epi } \rho_1\) for all \( n \in \mathbb{N} \) with \( Y_n \to Y \) in \( L^1 \) and \( r_n \to r \). Thus, for all \( n \in \mathbb{N} \) it holds \( \rho_1(Y_n) \leq r_n \). Since \( Y_n \to Y \) in \( L^1 \), there is a subsequence \( Y_{n_k} \) converging \( P \)-a.s. to \( Y \) (see Theorem 10.38 and 10.39 in [2]). The sequence \( L(-Y_{n_k}) \) converges \( P \)-a.s. to \( L(-Y) \) since \( l \) is continuous. \( L(-Y_{n_k}) \) is for all \( k \in \mathbb{N} \) a nonnegative random variable due to Assumption 4.18. Thus, we can apply Fatou’s Lemma (Lemma B.21) and obtain

\[
\forall Q \in \mathcal{Q} : \quad E^Q[L(-Y)] = E^Q[\liminf_{k \to \infty} L(-Y_{n_k})] \leq \liminf_{k \to \infty} E^Q[L(-Y_{n_k})]. \tag{4.32}
\]

Since \((Y_{n_k}, r_{n_k}) \in \text{epi } \rho_1\) for all \( k \in \mathbb{N} \), we have

\[
\forall Q \in \mathcal{Q} : \quad E^Q[L(-Y_{n_k})] \leq \sup_{\tilde{Q} \in \mathcal{Q}} E^{\tilde{Q}}[L(-Y_{n_k})] \leq r_{n_k}.
\]

Together with (4.32) we obtain

\[
\forall Q \in \mathcal{Q} : \quad E^Q[L(-Y)] \leq r.
\]

Hence, \( \rho_1(Y) = \sup_{Q \in \mathcal{Q}} E^Q[L(-Y)] \leq r \). This means, \((Y, r) \in \text{epi } \rho_1\) and thus \( \rho_1 \) is lower semicontinuous in \( L^1 \). \( \square \)

Since \( \rho_1 \) is monotone, we can apply Theorem 4.1 and obtain the static optimization problem that corresponds to the dynamic problem (4.31)

\[
\min_{\varphi \in R_0} \sup_{Q \in \mathcal{Q}} E^Q[L((1 - \varphi)H)], \tag{4.33}
\]

where \( R_0 \) is as in (4.5)

\[
R_0 = \{ \varphi : \Omega \to [0, 1], \mathcal{F}_T \text{ measurable, } \sup_{P^* \in P} E^{P^*}[\varphi H] \leq \tilde{V}_0 \}.
\]

By Proposition 4.21, \( \rho_1 \) is monotone, convex, lower semicontinuous and satisfies \( \rho_1(0) = 0 \). To apply Theorem 4.9, \( \rho_1 \) has to satisfy Assumption 4.5. Thus, \( \rho_1 \) has to be continuous and finite in some \((\varphi_0 - 1)H\) with \( \varphi_0 \in R_0 \). Since \( \rho_1 \) is a convex and lower semicontinuous functional on \( L^1 \), it is continuous in the interior of its effective domain ([11], Corollary I.2.5). The points, where \( \rho_1 \) takes
finite values, depend on the choice of the loss function \( l \). Let us consider the simple example of \( l(x) = x^2 \) and \( \mathcal{Q} = \{ P \} \). Let us ignore for the moment the condition \( l(x) = 0 \) for \( x \leq 0 \), which does not have an impact on the optimization problem (4.31) since we work only with nonnegative values. Then, the effective domain of the function \( \rho_1(Y) = E[Y^2] \) consists of all elements of \( L^2 \). Since the interior of \( L^2 \) as a linear subspace of \( L^1 \) is empty, there does not exist a point \( Y \in L^1 \) such that \( \rho_1(Y) = E[Y^2] \) is continuous. Thus, in general, we can not expect to find an inner point of the domain of \( \rho_1 \) and thus, a point where \( \rho_1 \) is continuous.

In the following, we shall consider several special cases, where the continuity of \( \rho_1 \) in some \( (\varphi_0 - 1)H \) with \( \varphi_0 \in \mathbb{R}_0 \) can be verified and thus, Theorem 4.9 can be applied to solve the problem.

The Special Case of Lipschitz Continuous Loss Functions

In this section, we consider a special case, i.e., we impose stronger assumptions to solve problem (4.31). These assumptions are for instance satisfied if the loss function is Lipschitz continuous and the set \( \mathcal{Q} \) of measures satisfies a certain condition. In addition to Assumption 4.20, we shall impose the following in this section.

**Assumption 4.22.** Let \( \{ Z_Q : Q \in \mathcal{Q} \} \subseteq L^\infty \) with \( \sup_{Q \in \mathcal{Q}} \| Z_Q \|_{L^\infty} < +\infty \).

Let \( \varepsilon > 0 \). We denote by \( U_\varepsilon(H) := \{ Y \in L^1 : \| Y - H \|_{L^1} \leq \varepsilon \} \) the \( \varepsilon \)-neighborhood of \( H \in L^1 \).

**Assumption 4.23.** Let \( l \) be such that there exists an \( \varepsilon \)-neighborhood \( U_\varepsilon(H) \) of \( H \) with

\[
\forall Y \in U_\varepsilon(H) : \quad L(Y) - L(H) \in L^1.
\]

**Remark 4.24.** Assumption 4.23 is for instance satisfied if \( l \) is Lipschitz continuous, i.e., there exists a constant \( c \in \mathbb{R} \) such that for all \( x, y \in \mathbb{R} \)

\[
|l(x) - l(y)| \leq c|x - y|.
\]

Then, it follows that \( L \) is Lipschitz continuous and maps into \( L^1 \) since for all \( Y_1, Y_2 \in L^1 \) we have

\[
\| L(Y_1) - L(Y_2) \|_{L^1} = \int_\Omega |L(Y_1)(\omega) - L(Y_2)(\omega)|dP = \int_\Omega |l(Y_1(\omega)) - l(Y_2(\omega))|dP \
\leq \int_\Omega c|Y_1(\omega) - Y_2(\omega)|dP = c\| Y_1 - Y_2 \|_{L^1}.
\]

**Remark 4.25.** If \( l \) is Lipschitz continuous and Assumption 4.22 is satisfied, then Assumption 4.20 holds since

\[
\sup_{Q \in \mathcal{Q}} E^Q[L(H)] \leq \| L(H) \|_{L^1} \sup_{Q \in \mathcal{Q}} \| Z_Q \|_{L^\infty} \leq c \| H \|_{L^1} \sup_{Q \in \mathcal{Q}} \| Z_Q \|_{L^\infty} < +\infty.
\]
4.1. **HEDGING IN COMPLETE AND SPECIAL INCOMPLETE MARKETS**

**Proposition 4.26.** Let Assumption 4.20, 4.22 and 4.23 be satisfied. Then, \( \rho_1 \) is continuous and finite in \(-H = (\varphi_0 - 1)H\) with \( \varphi_0 = 0 \in \mathbb{R}_0 \).

**Proof.** For all \(-Y \in U_{\epsilon}(H)\) we have \( L(-Y) - L(H) \in L^1 \) due to Assumption 4.23. Together with Assumption 4.22 and 4.20, we obtain that for all \(-Y \in U_{\epsilon}(H)\)

\[
\rho_1(Y) = \sup_{Q \in \mathcal{Q}} E^Q[L(-Y)] \leq \sup_{Q \in \mathcal{Q}} E^Q[L(-Y) - L(H)] + \sup_{Q \in \mathcal{Q}} E^Q[L(H)] \leq \|L(-Y) - L(H)\|_{L^1} \sup_{Q \in \mathcal{Q}} \|Z_Q\|_{L^\infty} + \sup_{Q \in \mathcal{Q}} E^Q[L(H)] < +\infty.
\]

Since \(-Y \in U_{\epsilon}(H)\) if and only if \( Y \in U_{\epsilon}(-H)\), we obtain that for all \( Y \in U_{\epsilon}(-H)\) the convex function \( \rho_1 \) is bounded above by a finite constant. Thus, by Lemma I.2.1, [11], \( \rho_1 \) is continuous in \(-H = (\varphi_0 - 1)H\) with \( \varphi_0 = 0 \in \mathbb{R}_0 \).

This means, if Assumption 4.18, 4.20, 4.22 and 4.23 are satisfied (for instance if we work with a Lipschitz continuous loss functions \( l \) and a finite set \( \mathcal{Q} \) with \( \{Z_Q : Q \in \mathcal{Q}\} \subseteq L^\infty \)), we can apply Theorem 4.9 to deduce a result about the structure of a solution \( \tilde{\varphi} \) to (4.33). Since \( \rho_1 : L^1 \rightarrow \mathbb{R} \cup \{+\infty\} \) is lower semicontinuous, convex, proper and monotone (Proposition 4.21), it has the following dual representation (see Theorem 1.5 (b))

\[
\rho_1(Y) = \sup_{Y^* \in L^\infty_+} \{E[-Y Y^*] - \rho_1^*(-Y^*)\}.
\]

In the following theorem we shall work with this dual representation instead of the representation (4.30) of \( \rho_1 \). The application of Theorem 4.9 yields the following result.

**Corollary 4.27** (Robust Efficient Hedging). Let Assumption 4.18, 4.20, 4.22 and 4.23 be satisfied and let \( Z_P \) be compact. Then, there exists a solution \( \tilde{\varphi} \) to (4.33). If \( \rho_1 \) is strictly convex, then any two solutions coincide \( P - a.s. \) on \( \{H > 0\} \).

Furthermore, there exists a pair \((\tilde{Y}^*, \tilde{\lambda}) \in L^\infty_+ \times \Lambda_+\) solving

\[
\max_{Y^* \in L^\infty_+, \lambda \in \Lambda_+} \left\{E[H Y^* \land H \int_P Z_P \cdot d\tilde{\lambda} - \tilde{V}_0 \lambda(Z_P) - \rho_1^*(-Y^*)]\right\}.
\]

(4.34)

Let \((\tilde{Y}^*, \tilde{\lambda})\) be the optimal pair in (4.34). It follows that:

- The solution of the static optimization problem (4.33) is

\[
\tilde{\varphi} = \begin{cases} 1 & : H \tilde{Y}^* > H \int_P Z_P \cdot d\tilde{\lambda} \\ 0 & : H \tilde{Y}^* < H \int_P Z_P \cdot d\tilde{\lambda} \end{cases} \quad P - a.s.
\]

with

\[
E^P[\tilde{\varphi} H] = \tilde{V}_0 \quad \tilde{\lambda} - a.s.
\]
• \((\tilde{\varphi}, \tilde{Y}^\ast)\) is a saddle point of the functional \((\varphi, Y^\ast) \mapsto E[(1 - \varphi)HY^\ast] - \rho^\ast(-Y^\ast)\) in \(R_0 \times L_\infty^\ast\).

• \((\tilde{V}_0, \tilde{\xi})\) solves the dynamic robust efficient hedging problem (4.31), where \(\tilde{\xi}\) is the superhedging strategy of the modified claim \(\tilde{\varphi}H\), obtained by the optional decomposition theorem (Theorem C.3).

Proof. The Assumptions 4.18, 4.20, 4.22 and 4.23 ensure that Assumption 4.5 is satisfied (see Proposition 4.21 and 4.26). Thus, we can apply Theorem 4.9 and the stated results follow. \(\square\)

Remark 4.28. The function \(\rho_1\) is strictly convex if for instance \(l(x)\) is strictly convex and \(\mathcal{Q}\) has only finitely many elements.

The Linear Case

Let us impose Assumption 4.22 for this section. Since the "linear" loss function \(l(x) = x^+\) is Lipschitz continuous, we can apply Corollary 4.27 (see Remark 4.24 and 4.25). But, in the linear case we can even go a step further.

Problem (4.33) with \(l(x) = x^+\) is equivalent to problem (4.33) with \(l(x) = x\) since we work only with nonnegative values. Thus, the static optimization problem in the linear case is

\[
\min_{\varphi \in R_0} \sup_{Q \in \mathcal{Q}} E^Q[(1 - \varphi)H], \tag{4.35}
\]

where \(R_0\) is as in (4.5) and the risk measure that is used is

\[
\rho_2(Y) = \sup_{Q \in \mathcal{Q}} E^Q[-Y],
\]

defined on \(L^1\). Since we impose Assumption 4.22, the risk measure \(\rho_2\) is a coherent risk measure on \(L^1\) (cf. Section 1.3) that is finite valued, thus continuous ([11], Corollary I.2.5). The maximal representing set of \(\rho_2\) is \(\mathcal{Q}_{\text{max}} = \overline{\text{co}}^* \mathcal{Q}\), the weak* closure of the convex hull of the densities of \(\mathcal{Q}\) (see Theorem 1.25). Then, the 0-1-structure of \(\tilde{\varphi}\) can be deduced with respect to a \(\tilde{Q} \in \overline{\text{co}}^* \mathcal{Q}\) instead of \(\tilde{Y}^\ast \in L_\infty^\ast\) as in Corollary 4.27. Thus, we obtain by an application of Corollary 4.17 with the maximal representing set \(\overline{\text{co}}^* \mathcal{Q}\) of \(\rho_2\) the following Corollary.

Corollary 4.29 (Robust Efficient Hedging with linear loss function). Let Assumption 4.22 be satisfied and let \(Z_P\) be compact. Then, there exists a solution \(\tilde{\varphi}\) to (4.35). Furthermore, there exists a pair \((\tilde{Q}, \tilde{\lambda})\) in \(\overline{\text{co}}^* \mathcal{Q} \times \Lambda_+\) solving

\[
\max_{Q \in \overline{\text{co}}^* \mathcal{Q}, \lambda \in \Lambda_+} \left\{ E[H Z_Q \wedge H \int_P Z_P \cdot d\lambda] - \tilde{V}_0 \lambda(Z_P) \right\}. \tag{4.36}
\]

Let \((\tilde{Q}, \tilde{\lambda})\) be the optimal pair in (4.36). It follows that:
4.1. HEDGING IN COMPLETE AND SPECIAL INCOMPLETE MARKETS

- The solution of the static optimization problem (4.35) is

\[
\tilde{\varphi} = \begin{cases} 
1 & : \ H\tilde{Z}_Q > H\int_P Z_P \, d\tilde{\lambda} \\
0 & : \ H\tilde{Z}_Q < H\int_P Z_P \, d\tilde{\lambda}
\end{cases} \quad P - \text{a.s.}
\]

with

\[E^P[\tilde{\varphi}H] = \tilde{V}_0 \quad \tilde{\lambda} - \text{a.s.}\]

- \((\tilde{\varphi}, \tilde{Q})\) is a saddle point of the functional \((\varphi, Q) \mapsto E^Q[(1 - \varphi)H]\) in \(R_0 \times \mathcal{P}^* Q\).

- \((\tilde{V}_0, \tilde{\xi})\) solves the dynamic problem (4.31) in the linear case, where \(\tilde{\xi}\) is the superhedging strategy of the modified claim \(\tilde{\varphi}H\), obtained by the optional decomposition theorem (Theorem C.3).

Corollary 4.29 is a generalization of Proposition 4.1 in Föllmer and Leukert [17] and a generalization of Theorem 1.19 in Xu [48] for the case \(Z_P\) compact. In [17] and [48] the set \(Q = \{P\}\) is a singleton. In [17], the problem is solved in the complete financial market, i.e., \(\mathcal{P} = \{P^*\}\) and in [48] the problem is solved in the incomplete financial market. Furthermore, in [48] the optimal strategy is computed in three complete market cases. In analogy to Nakano [31] (see Section 4.1.3), Xu [48] enlarged the set \(\mathcal{P}\) that contains the equivalent martingale measures and deduced the 0-1-structure of the optimal randomized test with respect to an element from the enlarged set. With our method this is not necessary, we work directly with the set \(\mathcal{P}\). Furthermore, we do not need to impose the assumption that the discounted asset price \(S\) is locally bounded as used in [48]. A comparison of the results in the general incomplete market can be found in Section 4.2.3.

With our method it is possible to solve the problem not only in the case \(Q = \{P\}\), but also for more general sets \(Q\) satisfying Assumption 4.22 and even for more general loss functions satisfying Assumption 4.23.

**Prospect: The General Case**

This section should be understood as a discussion and as a prospect of further research. The problem of robust efficient hedging does, in general, not satisfy Assumption 4.5. We shall show which results are affected and we give proposals how the problem could be solved.

Let us consider a general loss function \(l\) and problem (4.33) in the context of Chapter 2. All conditions of Assumption 2.1 are satisfied except the continuity of \(\rho_1\) in some \((\varphi_0 - 1)H\) with \(\varphi_0 \in R_0\) as postulated in (A5). The lack of this condition has an effect on the validity of strong duality in Theorem 2.6. Thus, the equality between the values of the primal problem (4.33) and its Fenchel dual problem is no longer ensured.
This motivates us to use the special structure of the problem and to define a modified risk measure on the space $L^\infty$ that might satisfy the required assumptions. First, we define $\tilde{L} : L^\infty \to L^1_+$ by

$$\tilde{L}(Y) := L(H - YH)$$

Consider the function $\rho : L^\infty \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\rho(Y) := \left\{ \begin{array}{ll} \sup_{Q \in \mathcal{Q}} E^Q[\tilde{L}(Y)] & : Y \in L^\infty_+ \\ +\infty & : Y \notin L^\infty_+ \end{array} \right.. \quad (4.37)$$

Then, problem (4.33) is equivalent to the static optimization problem

$$\min_{\varphi \in R_0} \rho(\varphi) \quad (4.38)$$

We shall deduce several properties of $\rho$.

**Proposition 4.30.** Suppose Assumption 4.18 and 4.20 hold. Then, the function $\rho$ defined in (4.37) is monotone, convex and proper with $\text{dom } \rho = L^\infty_+$. Furthermore, $\rho$ is lower semicontinuous and there exists a $\varphi_0 \in R_0$, such that $\rho$ is continuous and finite in $\varphi_0$.

**Proof.** $\rho$ is monotone and convex, since $l$ is nondecreasing and convex (Assumption 4.18). Take $Y \in L^\infty_+$. Then, since $l$ is nondecreasing, $H \in L^1_+$ (see page 50) and because of Assumption 4.20, $\rho(Y) = \sup_{Q \in \mathcal{Q}} E^Q[L(H - YH)] \leq \sup_{Q \in \mathcal{Q}} E^Q[L(H)] < +\infty$. For $Y \notin L^\infty_+$, we have $\rho(Y) = +\infty$. Thus, $\text{dom } \rho = L^\infty_+$ and $\rho$ is proper.

To show the lower semicontinuity of $\rho$, we shall show that the epigraph $\text{epi } \rho$ is closed. Take a sequence $(Y_n, r_n) \in \text{epi } \rho$ for all $n \in \mathbb{N}$ with $Y_n \to Y$ in the norm topology of $L^\infty$ and $r_n \to r$. Thus, for all $n \in \mathbb{N}$ we have $Y_n \in L^\infty_+$ with $\rho(Y_n) \leq r_n$. Then

$$\forall n \in \mathbb{N}, \forall Q \in \mathcal{Q} : \quad E^Q[\tilde{L}(Y_n)] \leq \sup_{Q \in \mathcal{Q}} E^Q[\tilde{L}(Y_n)] \leq r_n. \quad (4.39)$$

Take $Q \in \mathcal{Q}$. Since from $Y_n \to Y$ in the norm topology of $L^\infty$ it follows $Y_n \to Y$ $P$-a.s. (see Section 4.3 in [12]), we obtain that $\tilde{L}(Y_n)$ converges $P$-a.s. to $\tilde{L}(Y)$. Since for all $Y \in L^\infty_+$ it holds $0 \leq \tilde{L}(Y) \leq L(H)$ because of Assumption 4.18 and $H \in L^1_+$. It follows that $|\tilde{L}(Y_n)|$ is dominated by $L(H)$. Because of Assumption 4.20, $L(H)$ is integrable with respect to $Q \in \mathcal{Q}$. Thus, we can apply Corollary B.20 and obtain together with (4.39)

$$\forall Q \in \mathcal{Q} : \quad E^Q[\tilde{L}(Y)] = \lim_{n \to \infty} E^Q[\tilde{L}(Y_n)] \leq r. \quad (4.40)$$

Since $L^\infty_+$ is a closed set, $Y \in L^\infty_+$. It follows that $\rho(Y) = \sup_{Q \in \mathcal{Q}} E^Q[\tilde{L}(Y)] \leq r$. Thus, $(Y, r) \in \text{epi } \rho$, hence $\rho$ is lower semicontinuous.
4.1. HEDGING IN COMPLETE AND SPECIAL INCOMPLETE MARKETS

Since \( \rho \) is lower semicontinuous and convex and because of \( L^\infty \) endowed with the norm topology is a Banach space, \( \rho \) is continuous in the interior of its domain (see [11], Corollary 2.5). This means, \( \rho \) is continuous in \( \text{int}(L_+^\infty) \neq \emptyset \) (see Lemma B.13).

Take \( \varphi_0 \in \text{int}(L_+^\infty) \) with \( \varphi_0 \equiv c, \ c \in (0,1) \) such that \( cU_0 \leq \tilde{V}_0 \). Such a constant \( c \) always exists. Then \( \sup_{P \in \mathcal{P}} E^P [\varphi_0 H] \leq cU_0 \leq \tilde{V}_0 \). Thus, \( \varphi_0 \in R_0 \) and \( \rho \) is continuous and finite in \( \varphi_0 \).

**Proposition 4.31.** Suppose Assumption 4.18 and 4.20 hold. Then, the function \( \rho \) defined in (4.37) is lower semicontinuous with respect to the weak* topology.

**Proof.** The proof is similar to the proof of the lower semicontinuity of \( \rho \) in Proposition 4.30. We show that epi \( \rho \) is closed with respect to \( P-a.s. \) convergent sequences. Take a sequence \( \{(Y_n, r_n)\}_{n \in \mathbb{N}} \subset \text{epi} \rho \) with \( Y_n \rightarrow Y \) \( P-a.s \) and \( r_n \rightarrow r \). Thus, for all \( n \in \mathbb{N} \) it holds \( Y_n \in L_+^\infty \) with \( \rho(Y_n) \leq r_n \). Then

\[
\forall n \in \mathbb{N}, \forall Q \in \mathcal{Q} : \quad E^Q[\tilde{L}(Y_n)] \leq \sup_{Q \in \mathcal{Q}} E^Q[\tilde{L}(Y_n)] \leq r_n. \tag{4.40}
\]

Since \( \tilde{L}(Y_n) \) is dominated by the \( Q\)-integrable function \( L(H) \) for all \( n \in \mathbb{N} \), we can apply Corollary B.20 and obtain together with (4.40)

\[
\forall Q \in \mathcal{Q} : \quad E^Q[\tilde{L}(Y)] = \lim_{n \rightarrow \infty} E^Q[\tilde{L}(Y_n)] \leq r.
\]

Since \( L_+^\infty \) is closed with respect to \( P-a.s. \) convergent sequences, \( Y \in L_+^\infty \). Hence,

\[
\rho(Y) = \sup_{Q \in \mathcal{Q}} E^Q[\tilde{L}(Y)] \leq r.
\]

Thus, epi \( \rho \) is closed with respect to \( P-a.s. \) convergent sequences. Since \( \rho \) is convex, we can apply Theorem 1.7 and obtain that \( \rho \) is lower semicontinuous with respect to the weak* topology.

Since we work with the modified problem (4.38), we cannot apply Theorem 4.9 and have to work directly with the results of Chapter 2. The modified problem (4.38) turns out to be a special case of optimization problem (2.1) by setting \( A \varphi = \varphi, \ b = 0, \ C^* = Z_P, \ c = \tilde{V}_0 \) and \( X = \mathcal{Y} = L^\infty \), endowed with the norm topology. Hence, \( X^* = \mathcal{Y}^* = ba(\Omega, \mathcal{F}, P) \). \( H \in L_+^1 \) is the payoff of the contingent claim. It holds \( X_1 = R \) and \( X_0 = R_0 \). We check (A1)-(A7) of Assumption 2.1:

(A1): We have \( c = \tilde{V}_0 > 0 \) (see (4.3)).

(A2): It holds \( H \in L_+^1, C^* = Z_P \subseteq L^1 \) and \( \{HX^* : X^* \in C^*\} = \{HP^* : Z_P^* \in \mathcal{P}\} \subseteq L^1 \), since we assumed in (4.1) the superhedging price of \( H \) to be finite.

(A3): Inequality (4.1) also ensures that \( \sup_{X^* \in C^*} \|HX^*\|_{L^1} < +\infty \).
(A4): The operator $A : L^\infty \to L^\infty$, defined by $A \varphi := \varphi$, is linear and continuous.

(A5): Proposition 4.30 ensures the conditions for $\rho$.

(A6): The map $\varphi \mapsto \rho(A \varphi + b)$ coincides with $\rho(\varphi)$ and is lower semicontinuous in the weak* topology as proved in Proposition 4.31.

Note that Assumption (A7) is not satisfied, in general: The map $\varphi \mapsto \langle Y^*, A \varphi \rangle = \langle Y^*, \varphi \rangle$ is, in general, not continuous in the weak* topology for all $Y^* \in \text{dom} \rho^*$ since $\text{dom} \rho^* \subseteq ba(\Omega, \mathcal{F}, P)$.

**Remark 4.32.** One could think of endowing $\mathcal{Y} = L^\infty$ with the weak* topology since we already proved that $\rho$ is weakly* lower semicontinuous (Proposition 4.31). Then, Assumption (A7) is satisfied since $\text{dom} \rho^* \subseteq Y^* = L^1$. But in this case, we can not ensure the continuity of $\rho$ in $\varphi_0 \in R_0$ as postulated in (A5) since $\text{int} L^\infty_+ = \emptyset$ with respect to the weak* topology (Lemma B.14, cf. proof of Proposition 4.30).

It would be sufficient for the application of Theorem 2.9 to postulate that $\tilde{Y}^*$, the solution to the dual problem, is an element of $L^1$ and thus $\varphi \mapsto \langle \tilde{Y}^*, \varphi \rangle$ is continuous in the weak* topology. But, in general, this condition is not satisfied.

In the following, we show which results in the theorems of Chapter 2 do not longer hold since Assumption (A7) is not satisfied and we shall give proposals how the problem could be solved in spite of this. Assumption (A7) has no impact on Theorem 2.5 and 2.6.

Since $\rho$ is lower semicontinuous, convex and monotone with $\rho(0) < +\infty$ (Proposition 4.30), it has a dual representation with respect to elements of $Y^*_+ = ba(\Omega, \mathcal{F}, P)_+$ (see Theorem 1.5 (b)). Since $\rho$ is weakly* lower semicontinuous (Proposition 4.31), it is sufficient to consider elements of $L^1_+$ in the dual representation (see Theorem 1.6)

$$
\rho(Y) = \sup_{Y^* \in ba(\Omega, \mathcal{F}, P)_+} \{ \langle Y, -Y^* \rangle - \rho^* (-Y^*) \} = \sup_{Y^* \in L^1_+} \{ E[-YY^*] - \rho^* (-Y^*) \}.
$$

Then, the primal problem (4.38) can be written as

$$
\min_{\varphi \in R_0} \{ \sup_{Y^* \in L^1_+} \{ E[-\varphi Y^*] - \rho^* (-Y^*) \} \},
$$

where Theorem 2.5 ensures the existence of a primal solution $\tilde{\varphi}$. The continuity of $\rho$ in some $\varphi_0 \in R_0$ (Proposition 4.30) ensures strong duality between the primal and its dual problem (Theorem 2.6) with respect to $Y^* = ba(\Omega, \mathcal{F}, P)$, i.e.,

$$
\min_{\varphi \in R_0} \{ \sup_{Y^* \in L^1_+} \{ E[-\varphi Y^*] - \rho^* (-Y^*) \} \} = \sup_{Y^* \in ba(\Omega, \mathcal{F}, P)_+} \{ \inf_{\varphi \in R_0} \{ \langle \varphi, -Y^* \rangle - \rho^* (-Y^*) \} \}.
$$

Strong duality also ensures the existence of a dual solution $\tilde{Y}^* \in ba(\Omega, \mathcal{F}, P)_+$ and with equation (2.6) of Theorem 2.6 we obtain

$$
\min_{\varphi \in R_0} \{ \sup_{Y^* \in L^1_+} \{ E[-\varphi Y^*] - \rho^* (-Y^*) \} \} = \max_{Y^* \in ba(\Omega, \mathcal{F}, P)_+} \{ \min_{\varphi \in R_0} \{ \langle \varphi, -Y^* \rangle - \rho^* (-Y^*) \} \}.
$$

(4.41)
4.1. HEDGING IN COMPLETE AND SPECIAL INCOMPLETE MARKETS

In Theorem 2.8 we considered the inner problem of the dual problem for every \( Y^* \in \mathcal{Y}^* \), but it is sufficient to consider the inner problem just for \( \tilde{Y}^* \in ba(\Omega, \mathcal{F}, P)^+ \), i.e., the problem

\[
\max_{\varphi \in \mathbb{R}_0} \langle \varphi, \tilde{Y}^* \rangle, \tag{4.42}
\]

where (2.6) of Theorem 2.6 ensures the existence of a solution \( \tilde{\varphi} \). The dual problem of (4.42) is

\[
\inf_{\lambda \in \Lambda_+} \left\{ \sup_{\varphi \in \mathbb{R}} \langle \tilde{Y}^* - H \int P Z P^* d\lambda, \varphi \rangle - c\lambda(Z_P) \right\}, \tag{4.43}
\]

where \( \Lambda_+ \) is the set of all finite measures on \((Z_P, \mathcal{B})\) and \( \mathcal{B} \) is a \( \sigma \)-algebra of all Borel sets on \( Z_P \). Assumption (A7) does not have an impact on the validity of strong duality between (4.42) and (4.43) as in the proof of Theorem 2.8

\[
\max_{\varphi \in \mathbb{R}_0} \langle \varphi, \tilde{Y}^* \rangle = \min_{\lambda \in \Lambda_+} \left\{ \sup_{\varphi \in \mathbb{R}} \langle \tilde{Y}^* - H \int P Z P^* d\lambda, \varphi \rangle - c\lambda(Z_P) \right\}. \tag{4.44}
\]

The validity of strong duality also ensures the existence of a dual solution \( \tilde{\lambda} \in \Lambda_+ \) (Theorem A.12).

Assumption (A2) and (A3) ensure that \( H \int P Z P^* d\lambda \in L^1 \) and thus, the signed measure with density \( H \int P Z P^* d\lambda \) admits a Hahn decomposition for \( \lambda \in \Lambda_+ \). For simplicity, we write \( Y \) admits a Hahn decomposition instead of the measure with density \( Y \) admits a Hahn decomposition. With Assumption (A7) we wanted to ensure that \( \tilde{Y}^* \) is an element of \( L^1 \) and thus, the whole term \( \tilde{Y}^* - H \int P Z P^* d\lambda \) admits a Hahn decomposition which could be used in (4.44) (as it was in (2.22) in the proof of Theorem 2.8) and would lead to a result about the structure of a solution \( \tilde{\varphi} \) (see Theorem 2.9).

This makes clear that it is possible to weaken Assumption (A7) as proposed in Remark 2.3:

\((A7')\) \( A^* \tilde{Y}^* \) admits a Hahn decomposition.

At this point, we give several proposals of further research that could lead to a possibility to solve the modified problem (4.38) and thus, also the original problem (4.33) or that work directly with problem (4.33).

- **Does \( \tilde{Y}^* \) admit a Hahn decomposition?** \( \tilde{Y}^* \) is an element of \( ba(\Omega, \mathcal{F}, P)^+ \), a nonnegative, finitely additive set functions on \((\Omega, \mathcal{F})\) with bounded variation, absolutely continuous to \( P \) (see [49], Chapter IV, 9, Example 5). In [5] it was shown, that a bounded finitely additive real-valued measure \( \tilde{Y}^* \) admits a Hahn decomposition if and only if it attains its norm on the unit ball of \( L^\infty \). This is equivalent to the condition, that \( \tilde{Y}^* \) attains its bounds (see [40]). Schmidt [43], Lemma 2.1, showed that it is sufficient to show that the upper bound is
attained.
In (4.42) we see that \( \tilde{Y}^* \) attains its supremum over the set \( R_0 \) (follows from (2.6) of Theorem 2.6). If it also attains its supremum over the set \( R \), and thus over the unit ball in \( L^\infty \) (since \( \tilde{Y}^* \in ba(\Omega, \mathcal{F}, P)_+ \)), \( \tilde{Y}^* \) would admit a Hahn decomposition ([5], Theorem 1) and we could solve the problem analogously to Theorem 2.9.

**Approximation.** If \( \tilde{Y}^* \) does not admit a Hahn decomposition, \( \tilde{Y}^* \) could be approximated by a sequence \( \tilde{Y}^*_n \in ba(\Omega, \mathcal{F}, P) \), where \( \tilde{Y}^*_n \) admits a Hahn decomposition for all \( n \in \mathbb{N} \) or, if possible, even with a sequence \( \tilde{Y}^*_n \in L^1 \). For every \( \tilde{Y}^*_n \in L^1 \), there exists a solution \( \tilde{\varphi}_{\tilde{Y}^*_n} \) to

\[
\max_{\varphi \in R_0} \langle \varphi, \tilde{Y}^*_n \rangle,
\]

since \( R_0 \) is weakly* compact and the map \( \varphi \mapsto \langle \varphi, \tilde{Y}^*_n \rangle \) is weakly* continuous for \( \tilde{Y}^*_n \in L^1 \). Then, one has to analyze the behavior of the sequence \( \tilde{\varphi}_{\tilde{Y}^*_n} \) and to check if it converges to the solution \( \tilde{\varphi} \).

Furthermore, another possible approximation can be discussed. From (4.41) and Proposition 4.31 it follows that

\[
\rho(\tilde{\varphi}) = \langle \tilde{\varphi}, -\tilde{Y}^* \rangle - \rho^*(-\tilde{Y}^*) = \max_{Y^* \in ba(\Omega, \mathcal{F}, P)_+} \{ \langle \tilde{\varphi}, -Y^* \rangle - \rho^*(-Y^*) \}
\]

\[
= \sup_{Y^* \in L^1_+} \{ E[-\tilde{\varphi}Y^*] - \rho^*(-Y^*) \}.
\]

Hence, there exists a maximizing sequence \( Y^*_n \in L^1_+ \) such that \( E[-\tilde{\varphi}Y^*_n] - \rho^*(-Y^*_n) \) converges to \( \langle \tilde{\varphi}, -\tilde{Y}^* \rangle - \rho^*(-\tilde{Y}^*) \). Again, for every \( Y^*_n \in L^1 \), there exists a solution \( \tilde{\varphi}_{Y^*_n} \) to

\[
\max_{\varphi \in R_0} \langle \varphi, Y^*_n \rangle
\]

and one could analyze the behavior of the sequence \( \tilde{\varphi}_{Y^*_n} \).

**Is a weaker condition for strong duality satisfied?** Consider problem (4.33) and check if a weaker condition than the continuity in \((\varphi_0 - 1)H\) with \( \varphi_0 \in R_0 \) leads to a strong duality result (see [50], Theorem 2.7.1 for a list of conditions that lead to strong duality).

It seems to be worthwhile to do further research in this direction since for special cases there already exist results in the literature. In Föllmer and Leukert [17] the efficient hedging problem was considered. This is a special case of problem (4.31) with \( \mathcal{Q} = \{ P \} \), a singleton. For this case, the decomposition of the dynamic problem into the static problem and the representation problem (cf. Theorem 4.1) was
4.2. HEDGING IN INCOMPLETE MARKETS

already proved and the existence of a solution to the static problem was shown. Furthermore, a solutions for the linear case \( l(x) = x \) in the complete market \( \mathcal{P} = \{ P^* \} \) was deduced by an application of the Neyman-Pearson lemma. The linear case \( l(x) = x \) in the incomplete market was solved by Xu [48]. These special cases can be solved with the method deduced in this thesis as well (see also Section 4.2.3). It is even possible to solve the problem for the more general case of Lipschitz continuous loss functions and with a more general set \( \mathcal{Q} \) satisfying Assumption 4.22. Interesting for further research are more general loss functions, for instance the function \( l(x) = (x^+)^p, p \geq 1 \).

Kirch [26] considered the general robust efficient hedging problem (4.31) with the following assumptions concerning the loss function \( l \). In [26], \( l(x, \omega) \) was assumed to be strictly convex, increasing, continuous differentiable on \((0, H)\) and bounded for all \( x \geq 0 \). It is then possible to express the solution in terms of the inverse of the derivative of the utility function \( u := -\tilde{l} \). The problem was solved by enlarging the sets \( \mathcal{Q} \) and \( \mathcal{P} \) by passing to the closed convex hull of the densities of \( \mathcal{Q} \) in \( L^1 \) and to the closure of the densities of \( \mathcal{P} \) in \( L^0 \). The solution to the problem could be reduced to a solution to a simple problem (fixed \( Q \in \overline{\sigma Q} \) and fixed \( P^* \) in the closure of the densities of \( \mathcal{P} \)). In some cases only an approximation of the solution by a sequence of simple problems was possible. These results motivate further research in this area using the method deduced in this thesis.

4.2 Hedging in Incomplete Markets

In this section, we study the problem of hedging in incomplete markets in the general case, i.e., we only assume that the set of equivalent martingale measures satisfies \( \mathcal{P} \neq \emptyset \) due to absence of arbitrage opportunities. We do no longer impose compactness of \( Z_\mathcal{P} \). Let us consider problem (4.2), (4.3), i.e., the dynamic optimization problem of finding an admissible strategy that solves

\[
\min_{(\tilde{V}_0, \xi)} \rho \left( - (H - V_T)^+ \right), \quad 0 < \tilde{V}_0 < U_0.
\]  

(4.45)

We summarize the assumptions of this section.

**Assumption 4.33.**

- \( \mathcal{P} \neq \emptyset \).
- The payoff of the contingent claim satisfies \( H \in L^1_\mathcal{P} \).
- The superhedging price of \( H \) is finite, i.e., \( U_0 = \sup_{P^* \in \mathcal{P}} E^{P^*}[H] < +\infty \).

**Assumption 4.34.** The risk measure \( \rho : L^1 \to \mathbb{R} \cup \{ +\infty \} \) is a monotone, convex, lower semicontinuous function that is continuous and finite in some \( (\varphi_0 - 1)H \) with \( \varphi_0 \in \mathcal{R} \) and satisfies \( \rho(0) < +\infty \).
We can apply all theorems of Chapter 2 that do not need Assumption (A3) (compactness of $Z_P$). Since $\rho$ is monotone, we can apply Theorem 4.1 and obtain that the corresponding static optimization problem is

$$\min_{\varphi \in R_0} \rho ((\varphi - 1)H),$$

$$R_0 = \{ \varphi : \Omega \to [0,1], \mathcal{F}_T - \text{measurable}, \sup_{P^* \in P} E^{P^*} [\varphi H] \leq \tilde{V}_0 \}.$$ 

Theorem 2.5 ensures the existence of a solution $\tilde{\varphi}$ to (4.46) and the dual representation of $\rho$ (Theorem 1.5 (b)) enables us to rewrite (4.46) as follows

$$p = \min_{\varphi \in R_0} \rho ((\varphi - 1)H) = \min_{\varphi \in R_0} \{ \sup_{Y^* \in L_+} \{ E[(1 - \varphi)HY^*] - \rho^*(-Y^*) \} \}.$$ 

Theorem 2.6 ensures strong duality between (4.46) and its Fenchel dual problem

$$\sup_{Y^* \in L_+} \{ \inf_{\varphi \in R_0} \{ E[(1 - \varphi)HY^*] - \rho^*(-Y^*) \} \}$$

and ensures the existence of a saddle point, i.e.,

$$\min_{\varphi \in R_0} \{ \max_{Y^* \in L_+} \{ E[(1 - \varphi)HY^*] - \rho^*(-Y^*) \} \} = \max_{Y^* \in L_+} \{ \min_{\varphi \in R_0} \{ E[(1 - \varphi)HY^*] - \rho^*(-Y^*) \} \}.$$ 

It is no longer possible to solve the inner problem of the dual problem with the theory deduced in Chapter 2 (Theorem 2.8) since $Z_P$ is not assumed to be compact. In this section, we want to solve the inner problem with the method in Xu [48] that is based on a duality approach deduced by Kramkov and Schachermayer [27]. Let us consider the inner problem of the dual problem for a fixed $Y^* \in L_+$

$$\min_{\varphi \in R_0} E[(1 - \varphi)HY^*].$$

The existence of a solution $\tilde{\varphi}_{Y^*}$ to (4.48) follows from Lemma 2.7. Problem (4.48) is the static problem to the dynamic problem of finding an admissible strategy that minimizes

$$\min_{(V_0, \xi)} E[(H - V_T)^+ Y^*], \quad 0 < \tilde{V}_0 < U_0.$$ 

This problem was solved in [48] for the case $Y^* = 1$. We want to adopt the method to our case. Therefore, we introduce the set of admissible, self-financing value processes $V$ starting at initial capital $x > 0$

$$\mathcal{V}(x) := \{ V : V_t = x + \int_0^t \xi_s dS_s \geq 0, \quad t \in [0, T] \}$$

and the set of contingent claims super-replicable by some admissible self-financing strategies with initial capital $x$

$$\mathcal{C}(x) := \{ g \in L^0(\Omega, \mathcal{F}, P) : 0 \leq g \leq V_T \text{ for some } V \in \mathcal{V}(x) \}.$$
4.2. HEDGING IN INCOMPLETE MARKETS

We consider the state dependent utility function $U : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$

$$U(x, \omega) := H(\omega) Y^*(\omega) - (H(\omega) - x)^+ Y^*(\omega) = (H(\omega) \wedge x) Y^*(\omega)$$

(4.50)

and the primal problem for $x > 0$

$$u(x) = \sup_{V \in \mathcal{V}(x)} E[U(V_T(\omega), \omega)]$$

$$= \sup_{g \in \mathcal{C}(x)} E[U(g(\omega), \omega)] = \sup_{g \in \mathcal{C}(x)} E[(H \wedge g) Y^*].$$

(4.51)

If necessary, we use the notation $u_Y(\cdot)$ to emphasize the dependence from the selected $Y^* \in L^\infty$. Note that the problem $u(\tilde{V}_0)$ is equivalent to problem (4.49) in the sense that if $\tilde{g} \in \mathcal{C}(\tilde{V}_0)$ is a solution to (4.51) for $x = \tilde{V}_0$, then the admissible self-financing superhedging strategy $(\tilde{V}_0, \tilde{\xi})$ solves (4.49), where $\tilde{\xi}$ is obtained by the optional decomposition theorem (Theorem C.3). Furthermore, it holds (with (4.6)) that

$$-u(\tilde{V}_0) + E[HY^*] = \min_{(\tilde{V}_0, \tilde{\xi})} E[(H - V_T)^+ Y^*] = \min_{\varphi \in \mathcal{K}_0} E[(1 - \varphi) HY^*].$$

(4.52)

As in [48] and [27], we define the following set of processes $Y$

$$\mathcal{Y}(y) := \{ Y \geq 0 : Y_0 = y \text{ and } VY \text{ is a } P\text{-supermartingale for any } V \in \mathcal{V}(1) \}$$

and the set $\mathcal{D}(y)$ of random variables $h$ by

$$\mathcal{D}(y) := \{ h \in L^0(\Omega, \mathcal{F}, P) : 0 \leq h \leq Y_T \text{ for some } Y \in \mathcal{Y}(y) \}.$$

The dual relation between $\mathcal{C}(1)$ and $\mathcal{D}(1)$ (or equivalently between $\mathcal{V}(1)$ and $\mathcal{Y}(1)$) is for instance shown in [27], Proposition 3.1.

Let us consider the function $W : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ defined by

$$W(y, \omega) := \sup_{x \geq 0} \{ U(x, \omega) - xy \}$$

for $y \geq 0$. It holds $W(y, \omega) = (-U + \mathcal{I}_{\mathbb{R}_+})^*(-y, \omega)$ for each $\omega \in \Omega$ (see Definition A.3 for the definition of the conjugate function). With (4.50) and because of $W(0, \omega) \geq U(0, \omega) = 0$ we obtain

$$W(y, \omega) = (Y^*(\omega) - y)^+ H(\omega).$$

(4.53)

We assign to (4.51) the following dual problem

$$w(y) = \inf_{Y \in \mathcal{Y}(y)} E[W(Y_T(\omega), \omega)]$$

$$= \inf_{h \in \mathcal{D}(y)} E[W(h(\omega), \omega)] = \inf_{h \in \mathcal{D}(y)} E[(Y^* - h)^+ H].$$

(4.54)
The assumptions of Theorem C.7 are satisfied since (iii). It holds (cf. [48])

Proof. Let us consider the condition (ii) The value functions \( u \) and \( w \) are concave, continuous and increasing. The functions \( W(\cdot, \omega) \) and \( w \) are convex, continuous and decreasing. For the definition of the subdifferential of a convex function we refer to Definition A.13 and for that of a concave function we refer to Definition A.14. For fixed \( \omega \in \Omega \) we consider the function \( U : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) defined by \( U(g) := U(g(\omega), \omega) \) for \( g \in \mathcal{C}(x) \).

The subdifferential \( \partial U(g) \) is then understood for each \( \omega \in \Omega \) in the sense that \( h \in \partial U(g) \) \( P \)-a.s. Analogously we define the function \( W(h) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) for \( h \in \mathcal{D}(y) \) and the subdifferential \( \partial W(h) \). The following duality theorem holds true.

**Theorem 4.35.** Let Assumptions 4.33 be satisfied. Then, it holds:

(i) For \( x > 0 \) and \( y > 0 \) an optimal solution \( \tilde{g}(x) \in \mathcal{C}(x) \) to (4.51) exists and an optimal solution \( \tilde{h}(y) \in \mathcal{D}(y) \) to (4.54) exists.

(ii) The value functions \( u \) and \( w \) satisfy the following relationship

\[
\begin{align*}
  w(y) &= \sup_{x>0} \{ u(x) - xy \} \quad \text{for any } y > 0 \text{ and} \\
  u(x) &= \inf_{y>0} \{ w(y) + xy \} \quad \text{for any } x > 0.
\end{align*}
\]  

(iii) Let \( x > 0 \) and \( y > 0 \) such that \( y \in \partial u(x) \). Then, \( E[\tilde{g}\tilde{h}] = xy \) and \( \tilde{h} \in \partial U(\tilde{g}) \) \( P \)-a.s., or equivalently, \( \tilde{g} \in -\partial W(\tilde{h}) \) \( P \)-a.s. if and only if \( \tilde{g} \) solves (4.51) and \( \tilde{h} \) solves (4.54).

Proof. The assumptions of Theorem C.7 are satisfied since \( \mathcal{P} \neq \emptyset, U(\cdot, \omega) \) is continuous, increasing and concave for any fixed \( \omega \) and \( U(0, \omega) = 0 \). Furthermore, the right-hand derivative satisfies \( U^r(0, \omega) \geq 0 \) (cf. Remark C.8) and \( U^r(\infty, \omega) = \lim_{x \to -\infty} U^r(x, \omega) = 0 \). Since \( Y^* \in L\infty^\omega \) and \( H \in L^1_+ \), is holds \( U(x, \omega) \leq H(\omega)Y^*(\omega) \) \( P \)-a.s. for all \( x \geq 0 \) and \( HY^* \in L^1 \) since \( E[HY^*] \leq \|Y^*\|L\infty\|H\|L^1 < +\infty \). Then, the assertion of the theorem follows from Theorem C.7.

**Remark 4.36.** Note that the relationship in Theorem 4.35 (ii) means that \( w(y) = (-u + \mathcal{I}_{R_+})^*(y) \) and \( u(x) = -(w + \mathcal{I}_{R_+})^*(x) \).

Let us consider the condition \( x > 0 \) and \( y > 0 \) such that \( y \in \partial u(x) \) in Theorem 4.35 (iii). It holds (cf. [48])

\[ y \in \partial u(x) \iff u(x) = w(y) + xy, \]

which means that the infimum in (4.55) is attained.

For \( x > 0 \), we have \( \partial u(x) \neq \emptyset \) since \( u \) is continuous in the interior of its effective domain (see [11], Corollary I.2.3) and for all \( y \in \partial u(x) \) it holds \( y \geq 0 \).

The structure of a primal solution with respect to a dual solution can be deduced as follows.
4.2. Hedging in Incomplete Markets

**Theorem 4.37.** Let Assumption 4.33 be satisfied. Let \( x > 0 \) and \( y > 0 \) such that \( y \in \partial u(x) \). Let \( \tilde{h}(y) \in D(y) \) be an optimal solution to (4.54). Then, there is an optimal solution \( \tilde{g}(x) \) to (4.51) such that

\[
\tilde{g} = (1_{\{0 \leq \tilde{h} < Y^*\}} + \delta 1_{\{\tilde{h} = Y^*\}})H
\]

and

\[
E[\tilde{g}\tilde{h}] = xy,
\]

where \( \delta \) is a \([0, 1]\)-valued random variable.

**Proof.** Let \( x > 0 \) and \( y > 0 \) such that \( y \in \partial u(x) \). From Theorem 4.35 the existence of an optimal solution \( \tilde{g}(x) \in C(x) \) to (4.51) and \( \tilde{h}(y) \in D(y) \) to (4.54) follows. Furthermore, it holds \( E[\tilde{g}\tilde{h}] = xy \) and \( \tilde{g} \in -\partial W(\tilde{h}) \) \( P \)-a.s. It holds (cf. [48], page 8) that

\[
\tilde{g} \in -\partial W(\tilde{h}) \iff W(\tilde{h}) = U(\tilde{g}) - \tilde{g}\tilde{h}.
\]

With (4.50) and (4.53) this becomes

\[
\tilde{g} \in -\partial W(\tilde{h}) \iff (Y^* - \tilde{h})^+ H = (H \wedge \tilde{g})Y^* - \tilde{g}\tilde{h}.
\]

It follows

\[
-\partial W(\tilde{h}) = \begin{cases} 
0 & \text{if } \tilde{h} > Y^* \\
H & \text{if } 0 < \tilde{h} < Y^* \\
[H, \infty) & \text{if } \tilde{h} = 0 \\
[0, H] & \text{if } \tilde{h} = Y^*.
\end{cases}
\]

Thus, \( \tilde{g} = (1_{\{0 \leq \tilde{h} < Y^*\}} + \delta 1_{\{\tilde{h} = Y^*\}})H \in -\partial W(\tilde{h}) \) \( P \)-a.s. is an optimal solution to (4.51), where \( \delta \) is an \([0, 1]\)-valued random variable such that \( E[\tilde{g}\tilde{h}] = xy \) is satisfied.

To emphasize the dependence of the value function \( u \) and the solutions \( \tilde{g} \) and \( \tilde{h} \) from the selected \( Y^* \in L_+^{\infty} \), we use the notation \( u_{Y^*}, \tilde{g}_{Y^*}, \) and \( \tilde{h}_{Y^*} \).

Let the initial capital be \( x = \tilde{V}_0 \). We conclude that the optimal solution \( \tilde{g}_{Y^*}(\tilde{V}_0) \) to (4.51) can be written as \( \tilde{g}_{Y^*} = \tilde{g}_{Y^*} = (1_{\{0 \leq \tilde{h} < Y^*\}} + \delta 1_{\{\tilde{h} = Y^*\}}) \in R_0 \) is the solution to (4.48).

Now, we are ready to go back to the static optimization problem (4.46) and to deduce a result about the structure of its solution.

**Theorem 4.38 (Solution to the Generalized Hedging Problem).** Let Assumption 4.33 and 4.34 be satisfied. There exists a solution \( \tilde{g} \) to problem (4.46) and a solution \( \tilde{Y}^* \in L_+^{\infty} \) to problem (4.47).
Let $\tilde{y} > 0$ such that $\tilde{y} \in \partial u_{\tilde{y}}(\tilde{V}_0)$. Then, there exists an $\tilde{h} \in D(\tilde{y})$ such that the triple $(\tilde{Y}^*, \tilde{y}, \tilde{h}) \in (L^\infty_+ \times \mathbb{R}_{>0} \times D(y))$ solves
\[
\max_{Y^* \in L^\infty_+, y > 0, h \in D(y)} \{E[(Y^* \wedge h)H] - \tilde{V}_0y - \rho^*(-Y^*)\}.
\]

(4.56)

It follows that:

- The solution to (4.46) is
\[
\bar{\varphi} = \begin{cases} 
1 & : 0 \leq \tilde{h} < \tilde{Y}^* \\
0 & : \tilde{h} > \tilde{Y}^*
\end{cases}
P - \text{a.s.}
\]

with
\[
E[\bar{\varphi}H\tilde{h}] = \tilde{V}_0\tilde{y}.
\]

- $(\tilde{\varphi}, \tilde{Y}^*)$ is the saddle point of the functional $(\varphi, Y^*) \mapsto E[(1 - \varphi)HY^*] - \rho^*(-Y^*)$ in $R_0 \times L^\infty_+$.

- $(\tilde{V}_0, \tilde{\xi})$ solves the dynamic hedging problem (4.45), where $\tilde{\xi}$ is the superhedging strategy of the modified claim $\tilde{\varphi}H$, obtained by the optional decomposition theorem (Theorem C.3).

Proof. Theorem 2.5 ensures the existence of a solution $\tilde{\varphi}$ to (4.46). Consider the dual problem of (4.46) given in (4.47), where Theorem 2.6 ensures that the supremum with respect to $Y^* \in L^\infty_+$ and the infimum with respect to $\varphi \in R_0$ are attained. We obtain
\[
\max_{Y^* \in L^\infty_+} \min_{\varphi \in D} \{E[(1 - \varphi)HY^*] - \rho^*(-Y^*)\} \overset{(4.52)}{=} \max_{Y^* \in L^\infty_+} \{u_{Y^*}(\tilde{V}_0) + E[HY^*] - \rho^*(-Y^*)\}
\]
\[
\overset{(4.55)}{=} \max_{Y^* \in L^\infty_+} \{\min_{y > 0} \{w(y) + \tilde{V}_0y\} + E[HY^*] - \rho^*(-Y^*)\}
\]
\[
\overset{(4.54)}{=} \max_{Y^* \in L^\infty_+} \{\min_{y > 0} \{\min_{h \in D(y)} E[(Y^* - h)^+H] + \tilde{V}_0y\} + E[HY^*] - \rho^*(-Y^*)\}
\]
\[
= \max_{Y^* \in L^\infty_+, y > 0, h \in D(y)} \{E[(Y^* \wedge h)H] - \tilde{V}_0y - \rho^*(-Y^*)\}.
\]

With Theorem 2.6 it follows that $\tilde{Y}^*$ attains the maximum with respect to $Y^* \in L^\infty_+$. Remark 4.36 shows that the condition $\tilde{y} > 0$ such that $\tilde{y} \in \partial u_{\tilde{y}}(\tilde{V}_0)$ ensures that $\tilde{y}$ attains the above infimum with respect to $y > 0$. Theorem 4.35 shows that $\tilde{h} := \tilde{h}_{\tilde{y}, \tilde{V}_0}(\tilde{y}) \in D(\tilde{y})$ attains the above infimum with respect to $h \in D(\tilde{y})$. Thus, there exists a triple $(\tilde{Y}^*, \tilde{y}, \tilde{h}) \in (L^\infty_+ \times \mathbb{R}_{>0} \times D(y))$ solving (4.56). The application of Theorem 4.37 with $Y^* = \tilde{Y}^*$ leads to the result about the structure of $\tilde{\varphi}$. It follows that $(\tilde{\varphi}, \tilde{Y}^*)$ is the saddle point described in Theorem 2.6. Theorem 4.1 shows that $(\tilde{V}_0, \tilde{\xi})$ solves the dynamic hedging problem (4.45), where $\tilde{\xi}$ is the superhedging strategy of the modified claim $\tilde{\varphi}H$ obtained by the optional decomposition theorem (Theorem C.3).
4.2. Hedging in Incomplete Markets

If we compare Theorem 4.38 (general incomplete market) with Theorem 4.9 (complete and special incomplete markets), we see that both lead to a structural result of the solution \( \tilde{\varphi} \) to the static optimization problem (4.4). If \( Z_\mathcal{P} \) is compact, the 0-1-structure of an optimal randomized test \( \tilde{\varphi} \) can be deduced with elements from \( \mathcal{P} \) and elements from the representing set of the risk measure \( (L^\infty_+ \text{ or } \mathcal{Q}) \), this depends on the choice of the risk measure. In the general case, this is not possible any longer. The 0-1-structure of an optimal randomized test \( \tilde{\varphi} \) can be deduced as well with elements from the representing set \( L^\infty_+ \) (respectively \( \mathcal{Q} \)), but no longer with elements from \( \mathcal{P} \). We have to pass to a larger set \( D(y) \) which is a subset of \( L^0_+ \).

Thus, in the case where \( Z_\mathcal{P} \) is compact, we can deduce a more detailed result about the structure of \( \tilde{\varphi} \).

When special risk measures \( \rho \) as in Section 4.1 are considered, the results are analogously to Theorem 4.38. A special choice of \( \rho \) has an impact on the optimization problem (4.56) regarding the set, the solution \( \tilde{\varphi}^* \) is attained in, and on the representation of the conjugate function \( \rho^* \) of \( \rho \).

4.2.1 Convex Hedging

If \( \rho : L^1 \rightarrow \mathbb{R} \cup \{+\infty\} \) satisfies additionally to Assumption 4.34 the translation property, it forms a convex risk measure as in Assumption 4.12. Then, \( \rho \) admits the dual representation (see Theorem 1.16)

\[
\rho(Y) = \sup_{Q \in \mathcal{Q}} \{ E^Q[-Y] - \sup_{\tilde{Y} \in \mathcal{A}_\rho} E^Q[\tilde{Y}] \}, \tag{4.57}
\]

where \( \mathcal{Q} := \{ Q \in \hat{\mathcal{Q}} : Z_Q \in L^\infty_+ \} \) is the set of all probability measures \( Q \), absolutely continuous to \( P \) and with densities in \( L^\infty \) and \( \mathcal{A}_\rho \) is the acceptance set of \( \rho \). If we consider problem (4.46) with a convex risk measure \( \rho \) satisfying Assumption 4.12, its Fenchel dual problem (see Theorem 2.6) is

\[
\sup_{Q \in \mathcal{Q}} \inf_{\varphi \in \mathbb{R}_0^+} \{ E^Q[(1 - \varphi)H] - \sup_{Y \in \mathcal{A}_\rho} E^Q[-Y] \}. \tag{4.58}
\]

Then, Theorem 4.38 and the dual representation (4.57) of \( \rho \) lead to the following corollary.

**Corollary 4.39 (Convex Hedging).** Let Assumption 4.33 be satisfied and let \( \rho \) be a convex risk measure satisfying Assumption 4.12. There exists a solution \( \tilde{\varphi} \) to problem (4.46) and a solution \( \tilde{Q} \in \mathcal{Q} \) to (4.58). Let \( \tilde{y} > 0 \) such that \( \tilde{y} \in \partial u_{\tilde{\varphi}}(\tilde{V}_0) \).

Then, there exists an \( \tilde{h} \in D(\tilde{y}) \) such that the triple \( (\tilde{Q}, \tilde{y}, \tilde{h}) \in (\mathcal{Q} \times \mathbb{R}_0^+ \times D(\tilde{y})) \) solves

\[
\max_{Q \in \mathcal{Q}, y > 0, \tilde{h} \in D(\tilde{y})} \{ E[(Z_\mathcal{Q} \wedge h)H] - \tilde{V}_0 y - \sup_{Y \in \mathcal{A}_\rho} E^Q[-Y] \}.
\]

It follows that:
The solution to (4.46) is

\[ \tilde{\varphi} = \begin{cases} 
1 & : 0 \leq \tilde{h} < \tilde{Z}_Q \\
0 & : \tilde{h} > \tilde{Z}_Q 
\end{cases} \quad P - \text{a.s.} \]

with

\[ E[\tilde{\varphi} H \tilde{h}] = \tilde{V}_0 \tilde{y}. \]

*(\tilde{\varphi}, \tilde{Q}) is the saddle point of the functional \((\varphi, Q) \mapsto E^Q[(1 - \varphi) H - \sup_{Y \in \mathcal{A}} E^Q[-Y]] \in R_0 \times \mathcal{Q}.*

*(\tilde{V}_0, \tilde{\xi}) solves the dynamic hedging problem (4.45), where \(\tilde{\xi}\) is the superhedging strategy of the modified claim \(\tilde{\varphi} H\), obtained by the optional decomposition theorem (Theorem C.3).*

### 4.2.2 Coherent Hedging

Let us consider a coherent risk measure \(\rho : L^1 \to \mathbb{R} \cup \{+\infty\}\) satisfying Assumption 4.16. Then, the Fenchel dual problem of problem (4.46) is (see Theorem 2.6)

\[ \sup_{Q \in \mathcal{Q}} \inf_{\varphi \in R_0} E^Q[(1 - \varphi) H], \tag{4.59} \]

where \(\mathcal{Q}\), the maximal representing set of \(\rho\), is a convex and weakly* closed subset of \(\{Q \in \mathcal{Q} : Z_Q \in L^\infty\}\) (see Theorem 1.25). Theorem 4.38 and the dual representation (Theorem 1.25) of \(\rho\) lead to the following corollary.

**Corollary 4.40 (Coherent Hedging).** Let Assumption 4.33 be satisfied and let \(\rho\) be a coherent risk measure satisfying Assumption 4.16. There exists a solution \(\tilde{\varphi}\) to problem (4.46) and a solution \(\tilde{Q} \in \mathcal{Q}\) to problem (4.59). Let \(\tilde{y} > 0\) such that \(\tilde{y} \in \partial u_{\tilde{Q}}(\tilde{V}_0)\). Then, there exists an \(\tilde{h} \in \mathcal{D}(\tilde{y})\) such that the triple \((\tilde{Q}, \tilde{y}, \tilde{h}) \in (\mathcal{Q} \times \mathbb{R}_{>0} \times \mathcal{D}(y))\) solves

\[ \max_{Q \in \mathcal{Q}, y > 0, h \in \mathcal{D}(y)} \{E[(Z_Q \wedge h) H] - \tilde{V}_0 \tilde{y}\}. \]

It follows that:

* The solution to (4.46) is

\[ \tilde{\varphi} = \begin{cases} 
1 & : 0 \leq \tilde{h} < \tilde{Z}_Q \\
0 & : \tilde{h} > \tilde{Z}_Q 
\end{cases} \quad P - \text{a.s.} \]

with

\[ E[\tilde{\varphi} H \tilde{h}] = \tilde{V}_0 \tilde{y}. \]
4.2. HEDGING IN INCOMPLETE MARKETS

- \((\tilde{\varphi}, \tilde{Q})\) is the saddle point of the functional \((\varphi, Q) \mapsto E^Q[(1 - \varphi)H]\) in \(R_0 \times Q\).
- \((\tilde{V}_0, \tilde{\xi})\) solves the dynamic hedging problem (4.45), where \(\tilde{\xi}\) is the superhedging strategy of the modified claim \(\tilde{\varphi}H\), obtained by the optional decomposition theorem (Theorem C.3).

As in Section 4.1.3, we can compare Corollary 4.40 with the results of Nakano [32]. Corollary 4.40 shows that the typical 0-1-structure of an optimal randomized test \(\tilde{\varphi}\) is deduced with respect to elements from the sets \(Q\) and \(D(\tilde{y})\). Thus, with our method it is not necessary to consider the enlarged set \(Z\) that contains the set \(\{Z_Q : Q \in Q\}\) as in [32]. But in contrast to the complete case (Corollary 4.17) considered in Section 4.1, it is no longer possible to deduce the structure of \(\tilde{\varphi}\) directly with elements from \(Z_P\).

4.2.3 Robust Efficient Hedging

Let us consider a Lipschitz continuous loss function \(l\) satisfying Assumption 4.18. Let the risk measure in the problem of hedging in incomplete markets be the robust version of the expectation of the loss function (see Section 4.1.4)

\[
\rho_1(Y) = \sup_{Q \in \hat{Q}} E^Q[L(Y)], \quad Y \in L^1,
\]

where \(L : L^1 \to L^0_+\) is as in Section 4.1.4 defined by \(L(Y)(\omega) := l(Y(\omega))\). The probability measures \(Q \subseteq \hat{Q}\) take into account an uncertainty regarding the underlying model and satisfy Assumption 4.22. We can show that the risk measure \(\rho_1\) satisfies Assumption 4.34 (see Proposition 4.21 and 4.26) and has the dual representation

\[
\rho_1(Y) = \sup_{Y^* \in L^*_+} \{E[-YY^*] - \rho_1^*(-Y^*)\}.
\]

Thus, this fits exactly into the setting of Theorem 4.38 and we can solve the problem by an application of this theorem. Analogously, we can treat the more general case where the loss function \(l\) and the set of probability measures \(Q\) satisfy Assumption 4.18, 4.20, 4.22 and 4.23.

In the case of a linear loss function, we can go a step further and deduce the structure of the solution \(\tilde{\varphi}\) with respect to elements from \(Q\). We consider the hedging problem (4.46) with the risk measure

\[
\rho_2(Y) = \sup_{Q \in \hat{Q}} E^Q[-Y], \quad (4.60)
\]

which is a continuous coherent risk measure on \(L^1\) with the maximal representing set \(Q_{\text{max}} = \overline{Q}\) (see Section 4.1.4). Its Fenchel dual problem is

\[
\sup_{Q \in \overline{Q}} \inf_{Q \in R_0} E^Q[(1 - \varphi)H]. \quad (4.61)
\]
Corollary 4.41 (Robust Efficient Hedging with linear loss function). Let the risk measure $\rho$ be as in (4.60) and let Assumption 4.22 and 4.33 be satisfied. There exists a solution $\tilde{\varphi}$ to problem (4.46) and a solution $\tilde{Q} \in \overline{\mathcal{Q}}^*$ to its dual problem (4.61). Let $\tilde{y} > 0$ such that $\tilde{y} \in \partial u_{\tilde{Q}}(\tilde{V}_0)$. Then, there exists an $\tilde{h} \in \mathcal{D}(\tilde{y})$ such that the triple $(\tilde{Q}, \tilde{y}, \tilde{h}) \in (\overline{\mathcal{Q}}^* \times \mathbb{R}_+ \times \mathcal{D}(y))$ solves

$$
\max_{Q \in \overline{\mathcal{Q}}^*} \{E[(Z_{\bar{Q}} \wedge \bar{h})H] - \tilde{V}_0 \tilde{y}\}.
$$

It follows that:

- The solution to (4.46) is

$$
\tilde{\varphi} = \begin{cases} 
1 & : \ 0 \leq \tilde{h} < \tilde{Z}_{\tilde{Q}} \\
0 & : \ \tilde{h} > \tilde{Z}_{\tilde{Q}}
\end{cases} 
\quad P - \text{a.s.}
$$

with

$$
E[\tilde{\varphi} H \tilde{h}] = \tilde{V}_0 \tilde{y}.
$$

- $(\tilde{\varphi}, \tilde{Q})$ is the saddle point of the functional $(\varphi, Q) \mapsto E^Q[(1-\varphi)H]$ in $R_0 \times \overline{\mathcal{Q}}^*$.

- $(\tilde{V}_0, \tilde{\xi})$ solves the dynamic hedging problem (4.45), where $\tilde{\xi}$ is the superhedging strategy of the modified claim $\tilde{\varphi}H$, obtained by the optional decomposition theorem (Theorem C.3).

With Theorem 4.38, the robust efficient hedging problem can be solved when the loss function $l$ and the set of probability measures $\mathcal{Q}$ satisfy Assumption 4.18, 4.20, 4.22 and 4.23. Corollary 4.41 treats the special case of a linear loss function. These results generalize Proposition 4.1 in Föllmer and Leukert [17] and Theorem 1.19 in Xu [48]. In [17] and [48] the set $\mathcal{Q} = \{P\}$ is a singleton and a linear loss function is considered. In [17], the problem is solved in the complete financial market, i.e., $\mathcal{P} = \{P^\ast\}$ and in [48] the problem is solved in the incomplete financial market. With our method it is possible to solve the problem not only in the case $\mathcal{Q} = \{P\}$, but also for more general sets $\mathcal{Q}$ satisfying Assumption 4.22 and even for more general loss functions.

Example 4.42. For a risk measure $\rho$ (regardless if it is a convex or coherent risk measure or as general as in Assumption 4.5), the structure of a solution $\tilde{\varphi}$ to the hedging problem (4.2), (4.3) is not given explicitly (it depends on the dual solutions $(\tilde{Y}^*, \tilde{\lambda})$ in the case where $Z_P$ is compact (see Theorem 4.9), respectively on the dual solutions $(\tilde{Y}^*, \tilde{y}, \tilde{h})$ in the general incomplete market (see Theorem 4.38)). We give
4.2. HEDGING IN INCOMPLETE MARKETS

a very simple example that is connected to different kinds of risk measures and a special case, where the problem can be solved explicitly.

Let us consider the problem of minimizing the risk of losses $-(H - V_T)^+$ due to the shortfall where the risk is measured by the coherent risk measure $\rho : L^1 \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\rho(X) = E^Q[-X].$$

This means, the representing set in the dual representation of the coherent risk measure (see Theorem 1.25) is a singleton, $Q = \{Q\}$ with $Z_Q \in L^\infty$. Thus we look for an admissible strategy $(V_0, \tilde{\xi})$ that minimizes

$$\rho(-(H - V_T)^+) = E^Q[(H - V_T)^+] \quad (4.62)$$

under the constraint

$$0 < V_0 \leq \tilde{V}_0, \quad (4.63)$$

where $\tilde{V}_0$ is a given capital constraint that is strictly less than the superhedging price $U_0$ of $H$. Theorem 4.1 shows that the corresponding static optimization problem is

$$\max_{\varphi \in \mathbb{R}} E^Q[\varphi H] \quad (4.64)$$

under the constraint

$$\forall P^* \in \mathcal{P} : E^{P^*}[\varphi H] \leq \tilde{V}_0. \quad (4.65)$$

The same optimization problem with $HZ_Q = Z_P$ arises in [16], Section 4, where the problem of quantile hedging in the incomplete case is considered. The risk measure used there is just the probability of the shortfall.

In [17], the expectation of a loss function is used as a risk measure. In Section 4, the problem of minimizing the expected shortfall is considered. This means, the linear loss function $l(x) = x$ is used. This leads to the optimization problem (4.64), (4.65) with $Q = P$.

Corollary 4.17 and 4.40 make it possible to solve these problems not only in the complete market. We consider two cases. First, let $Z_P$ be compact. Under the assumption $\tilde{V}_0 > 0$, the following conditions are necessary and sufficient for the optimality of $\tilde{\varphi}$ with respect to the optimization problem (4.64), (4.65) and give a result about the structure of the solution (see Corollary 4.17):

$$\tilde{\varphi} = \begin{cases} 1 & : \quad HZ_Q > H \int_P Z_P \cdot d\tilde{\lambda} \\ 0 & : \quad HZ_Q < H \int_P Z_P \cdot d\tilde{\lambda} \end{cases} \quad P - a.s.$$

with

$$E^{P^*}[\tilde{\varphi}H] = \tilde{V}_0 \quad \tilde{\lambda} - a.s.,$$
where \( \tilde{\lambda} \), a finite measure on \( \mathcal{P} \), is the solution of the dual problem of (4.64), (4.65), i.e., a solution to
\[
\inf_{\lambda \in \Lambda_+} \left\{ E[(HZ_Q - H \int P Z_p d\lambda)^+] + \tilde{V}_0 \lambda(Z_P) \right\}.
\]

In the general incomplete market, we can apply Corollary 4.40 and obtain that the structure of the optimal solution \( \tilde{\varphi} \) of (4.64), (4.65) is the following
\[
\tilde{\varphi} = \begin{cases} 
1 & : \ 0 \leq \tilde{h} < Z_Q \\
0 & : \ \tilde{h} > Z_Q 
\end{cases} \quad P - \text{a.s.}
\]
with
\[
E[\tilde{\varphi} H \tilde{h}] = \tilde{V}_0 \tilde{y},
\]
where \( \tilde{y} \in \partial u_Q(\tilde{V}_0) \) is assumed to satisfy \( \tilde{y} > 0 \) and \( \tilde{h} \in D(\tilde{y}) \) solves
\[
\inf_{h \in D(\tilde{y})} E[(Z_Q - h)^+].
\]

In both cases, the dynamic coherent hedging problem (4.62), (4.63) can be solved by the optional decomposition theorem (Theorem C.3). The solution is \((\tilde{V}_0, \tilde{\xi})\), where \( \tilde{\xi} \) is the superhedging strategy of the corresponding modified claim \( \tilde{\varphi} H \) (see Theorem 4.1).

If additionally \( \mathcal{P} = \{P^*\} \) is a singleton as in [32], Proposition 4.1, i.e., we work in a complete financial market, but with capital constraint \( \tilde{V}_0 < U_0 = E^{P^*}[H] \) and can apply Corollary 4.17. Then, the static problem can be solved explicitly. The optimal solution is
\[
\tilde{\varphi}(\omega) = 1_{\{Z_Q > \tilde{a}Z_{P^*}\}}(\omega) + \delta 1_{\{Z_Q = \tilde{a}Z_{P^*}\}}(\omega),
\]
where
\[
\tilde{a} = \inf\{a \mid E^{P^*}[H1_{\{Z_Q > aZ_{P^*}\}}] \leq \tilde{V}_0\}
\]
and
\[
\delta = \left\{ \begin{array}{cl}
\frac{\tilde{V}_0 - E^{P^*}[H1_{\{Z_Q > \tilde{a}Z_{P^*}\}}]}{E^{P^*}[H1_{\{Z_Q = \tilde{a}Z_{P^*}\}}]} & : \ P^*(\{Z_Q = \tilde{a}Z_{P^*}\} \cap \{H > 0\}) > 0 \\
c \in [0, 1] & \text{arbitrarily} \quad : \ P^*(\{Z_Q = \tilde{a}Z_{P^*}\} \cap \{H > 0\}) = 0.
\end{array} \right.
\]

When \( Q \) is equal to \( P \) this coincides with Proposition 4.1 in [17]. If there would be no capital constraint in the complete case, the optimal randomized test of the static problem would be \( \tilde{\varphi} = 1 \) on \( \{H > 0\} \). That means \( \tilde{\varphi} H = H \). Thus, the optimal strategy of problem (4.62), (4.63) would be exactly the replicating strategy \((E^{P^*}[H], \tilde{\xi})\) of the claim \( H \).