Chapter I

The Portfolio optimization problem: A general overview

2 Financial markets

We consider a continuous-time economy with finite horizon $[0, T]$ which is built on a complete probability space $(\Omega, \mathcal{H}, P)$, on which is defined an $n$-dimensional Brownian motion $W = (W_t)_{t \in [0,T]}$, $W_t = (W^1_t, \ldots, W^n_t)^\top$. We shall denote by $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}$ the $P$-augmentation of the natural filtration and assume that $\mathcal{H} = \mathcal{F}_T$. Further, we assume that all stochastic processes are adapted to $\mathcal{F}$. It is assumed through this thesis that all inequalities as well as equalities hold $P$-almost surely. Moreover, it is assumed that all stated processes are well defined without giving any regularity conditions ensuring this. As in the Black-Scholes model [9], financial investment opportunities are given by an instantaneously risk-free market account providing an interest rate $r$ so that its price $S^0 = (S^0_t)_{t \in [0,T]}$ is given by

$$dS^0_t = rS^0_t \, dt,$$  \hspace{1cm} (2.1)$$

and $n$ risky stocks whose prices $S = (S_t)_{t \in [0,T]}$, $S_t = (S^1_t, \ldots, S^n_t)^\top$ evolve according to

$$dS^i_t = S^i_t \left[ \mu_i \, dt + \sum_{j=1}^n \sigma_{ij} dW^j_t \right], \quad S^i_0 = s^i_0 \in \mathbb{R}, \quad i = 1, \ldots, n,$$  \hspace{1cm} (2.2)$$

where the interest rate $r$, the stock instantaneous mean return $\mu = (\mu_1, \ldots, \mu_n)^\top$ and the volatility matrix $\sigma = (\sigma_{ij})_{i,j=1,\ldots,n}$ may be time-depending, but defined in a way to insure the integrability conditions. Trading in the this economy requires from an investor which we shall call portfolio manager

- an initial capital $x > 0$,
- a portfolio process $\theta = (\theta_t)_{t \in [0,T]}$, $\theta_t = (\theta^1_t, \ldots, \theta^n_t)^\top$ where $\theta^i_t$ indicates the fraction of wealth invested in stock $i$ at time $t$,  

• a consumption process $c_t = (c_t)_{t \in [0,T]}$ where $c_t$ indicates the rate with which the wealth generated by the portfolio $\theta$ is consumed at time $t$.

Depending on the financial conditions of the investor one can impose to the portfolio and the consumption processes appropriate conditions to determine the set of admissible strategies. A classical example would be that strategies taking values only in the interval $[0,1]$, i.e., there is no short-selling.

At any time $t$, a portfolio manager acts in this market by choosing a trading strategy $\psi = (\psi_t)_{t \in [0,T]}$, $\psi_t = (\psi_1^t, \ldots, \psi_n^t)^\top$ where $\psi_i^t$ is the number of shares held by the portfolio manager in the asset $i$. The $\mathbb{R}^{n+1}$-valued process $\psi$ is assumed to be $\mathcal{F}$-measurable such that

$$\sum_{i=0}^{n} \int_0^T (\psi_i^t S_i^t)^2 dt < \infty.$$ 

The wealth process $X_t$ of the portfolio manager is defined at time $t$ in terms of the trading strategy by

$$X_t = \sum_{i=0}^{n} \psi_i^t S_i^t.$$

Moreover, we consider that the trading strategy is self-financing in the sense that no other money is going in or out the market except the money generated by the trading strategy, see [49]. Under this assumption and if the wealth $X_t > 0$, $P - a.s$, the portfolio manager can act in the market using the associated portfolio process $\theta = (\theta_t)_{t \in [0,T]}$, $\theta_t = (\theta_1^t, \ldots, \theta_n^t)^\top$ defined as the fractions of wealth invested in each risky stock, i.e.,

$$\theta_i^t = \frac{\psi_i^t S_i^t}{X_t^\theta}, \quad i = 1, \ldots, n,$$

with $\theta_0^t = 1 - \sum_{i=1}^{n} \theta_i^t$ is the fraction of wealth invested in the risk-free bond. As a consequence, the wealth process can be formulated in terms of the portfolio process as a linear stochastic differential equation given by

$$dX_t^\theta = X_t^\theta \left( r \left( 1 - \sum_{i=1}^{n} \theta_i^t \right) + \sum_{i=1}^{n} \mu_i \theta_i^t \right) dt + X_t^\theta \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \theta_i^t dW_j^t \right).$$

Using matrix notation, the dynamics of the wealth process can be formulated as

$$dX_t^\theta = X_t^\theta [r + \theta_1^t (\mu - r 1_n)] dt + X_t^\theta \theta_1^t \sigma dW_t, \quad X_0^\theta = x. \quad (2.3)$$

**Remark 2.1.** In Chapter IV, we shall use another equivalent quantity for the investment decision which is the amount of wealth invested in risky stocks. More precisely, we define the amount of wealth invested in the risky stocks by $\pi_t = (\pi_t)_{t \in [0,T]}$, $\pi_t = (\pi_1^t, \ldots, \pi_n^t)^\top$, where $\pi_i^t = \psi_i^t S_i^t$ indicates the amount of wealth invested in stock $i$ at time $t$. Further, $\pi_i^0 = X_t - \sum_{i=1}^{n} \pi_i^t$ is the amount of wealth invested in the risk-free bond. As consequence the wealth process can be reformulated in terms of the process $\pi$ as

$$dX_t^\pi = \pi_1^t (\mu dt + \sigma dW_t) + (X_t^\pi - 1_n \pi_t) r dt, \quad X_0^\pi = x. \quad (2.4)$$
3 Utility function of an investor

In his exposition about the theory on the measurement of risk Bernoulli [6] proposed that the value of an item should not be determined by the price somebody has to pay for it, but by the utility that this item has for the owner. A classical example would be that a glass of water has a much higher utility for somebody who is lost in the desert than somebody in the civilization. Although the glass of water might be exactly the same and therefore its price, the two persons in the mentioned situation will perceive its value differently.

Following partially the structure given in [22] Chapter 10, we discuss different properties that a given utility function should have and we look at some typical economic utility functions. Intuitively everybody prefers more wealth $X$ than less wealth and this is the first property we are interested to get for the utility function. Economists call this the non-satiation attribute and it expresses the fact that an investment with higher return has always a higher utility than an investment with a lower return assuming that both options equally likely.

From this it seems to be important to focus on a utility function whose first derivative has to be positive. Hence, the first requirement placed on a utility function $U$ from a wealth parameter $X$ is therefore

$$U'(X) > 0.$$ 

The second property of a utility function is an assumption about an investor’s taste of risk. Three assumptions are possible: The investor is averse to risk, the investor is neutral toward risk and the investor seeks risk. A simple example illustrating the three different risks is the so called fair gamble which is an investment with expected value equal to its cost.

<table>
<thead>
<tr>
<th>Invest</th>
<th>Do Not Invest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outcome</td>
<td>Probability</td>
</tr>
<tr>
<td>2</td>
<td>1/2</td>
</tr>
<tr>
<td>0</td>
<td>1/2</td>
</tr>
</tbody>
</table>

The option “invest” has an expected value of $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$ Euro. Assume that the investor would have to pay 1 Euro to undertake this investment and obtain these outcomes. If the investor prefers to not invest, then 1 Euro is kept. In this case the expected utility of not investing must be higher than the expected utility of investing

$$U(1) > \frac{1}{2} U(2) + \frac{1}{2} U(0),$$

which implies that

$$U(1) - U(0) > U(2) - U(1),$$

and this inequality expresses that the change in utility when wealth changes from 0 to 1 is more valuable than the change in utility when wealth changes from 1 to 2 and hence the utility is modeled by a function such that an additional unit increase is...
less valuable than the last unit increase. This property is fulfilled for a function with negative second derivative.

Risk neutrality means that an investor is indifferent to weather or not a fair gamble option is undertaken. When the investor is indifferent between investing and not investing, the expected utility of investing, or not investing, must be the same

\[ U(1) = \frac{1}{2} U(2) + \frac{1}{2} U(0), \]

and this yields after rearranging

\[ U(1) - U(0) = U(2) - U(1). \]

This expression implies that the change in utility of wealth is independent of the changes in wealth. Such a characteristic is fulfilled for functions that exhibit a zero second derivative, therefore indifference to a fair gamble implies a utility function that should have a zero second derivative.

Risk seeking means that the investor would select a fair gamble and hence the expected utility of investment must be higher than the expected utility of not investing

\[ U(1) < \frac{1}{2} U(2) + \frac{1}{2} U(0), \]

and this yields after rearranging

\[ U(1) - U(0) < U(2) - U(1). \]

This inequality expresses that the utility of one-unit change from 2 to 1 is greater than the utility of one-unit change from 0 to 1. Functions with positive second derivative are functions that exhibit the property of greater change in value for larger unit changes in the argument.

Another property of utility functions can be derived from the example of an investor who is deriving some utility from the wealth obtained by investing in risky assets and risk-free bound. Here, the property is an assumption about how the size of the wealth invested in risky assets changes when the size of the wealth has changed. In other words, if the investor’s wealth increases, will be more or less of that wealth invested in risky stock? Here three kinds of investor’s behavior are possible:

- Decreasing absolute risk aversion: The investor increases the amount invested in risky assets when the wealth increases.
- Constant absolute risk aversion: The investor keeps the amount invested in risky assets when the wealth increases.
- Increasing absolute risk aversion: The investor decreases the amount invested in risky assets when the wealth increases.
As it is pointed out in [22], the absolute risk aversion can be measured by

\[ A(X) = -\frac{U''(X)}{U'(X)}, \]

and as a consequence the type of the investor can be determined according to \( A'(X): \)

- \( A'(X) > 0 \): Increasing absolute risk aversion
- \( A'(X) = 0 \): Constant absolute risk aversion
- \( A'(X) < 0 \): Decreasing absolute risk aversion.

It is also possible to use the change of the percentage of wealth invested in risky assets as wealth changes. This is evaluated by

\[ R(X) = -\frac{XU''(X)}{U'(X)} = XA(X), \]

and has the following interpretation

- \( R'(X) > 0 \): Increasing relative risk aversion
- \( R'(X) = 0 \): Constant relative risk aversion
- \( R'(X) < 0 \): Decreasing relative risk aversion.

The most frequently used utility function is the power utility function

\[ U(z) = \left\{ \begin{array}{ll}
\frac{z^{1-\gamma}}{1-\gamma}, & \gamma \in (0, \infty) \setminus \{1\}; \\
\ln z, & \gamma = 1.
\end{array} \right. \tag{3.5} \]

With positive first derivative and negative second derivative, the power utility function (3.5) meets the requirement of risk averse investor who prefers more than less wealth. Moreover, this function exhibits decreasing absolute risk aversion and constant relative risk aversion. The parameter \( \gamma \) of the power utility function can be interpreted as relative risk aversion since in this case \( R(X) = \gamma \).

**Remark 3.1.** If we choose a logarithmic utility function \( U(z) = \ln z \), i.e., we set \( \gamma = 1 \), then the utility \( U(X) \) of the terminal wealth \( X \) is equivalent to the annual logarithmic return. The annual logarithmic return is defined as

\[ L(X) := \frac{1}{T} \ln \frac{X}{x}, \]

where \( x \) is the initial capital, i.e., \( X_0 = x \).
4 The portfolio optimization problem

4.1 Pointwise maximization

In this section we examine the particular case of portfolio optimization when the investor maximizes the expected logarithmic utility of the terminal wealth of one stock with a constant stock return $\mu$ and a constant volatility $\sigma$, and a bond with a constant interest rate $r$. The portfolio manager starts with initial capital $x > 0$ and follows a trading strategy $\theta = (\theta_t)_{t \in [0,T]}$ which leads to the wealth $X^\theta = (X^\theta_t)_{t \in [0,T]}$ defined by

$$dX^\theta_t = X^\theta_t[(r + \theta_t(\mu - r))dt + \theta_t\sigma dW_t], \quad X^\theta_0 = x,$$

which can be expressed as

$$X^\theta_t = x \exp\left(\int_0^t \left(r + \theta_s(\mu - r) - \frac{1}{2}(\theta_s\sigma)^2\right)ds + \int_0^t \theta_s\sigma dW_s\right).$$

The strategy $\theta$ is required to be such that the stochastic integral $(\int_0^t \theta_s\sigma dW_s)_{t \in [0,T]}$ is a martingale, which is the case when the inequality $E\left[\int_0^T \theta^2_s ds\right] < \infty$ is fulfilled, or when $\theta$ is assumed to be a bounded and deterministic. Moreover, the particular structure of the logarithmic utility allows to get

$$E[\ln(X^\theta_T)] = \ln(x) + E\left[\int_0^t \left(r + \theta_s(\mu - r) - \frac{1}{2}(\theta_s\sigma)^2\right)ds + \int_0^t \theta_s\sigma dW_s\right],$$

The portfolio maximization is now equivalent to the following pointwise maximization problem

find a strategy $\theta^*$ which maximizes $\left(r + \theta_t(\mu - r) - \frac{1}{2}(\theta_t\sigma)^2\right)$.

Here, the optimal solution is given by a constant strategy

$$\theta^*_t = \theta^*, \quad \text{for all} \quad 0 \leq t \leq T \quad \text{with} \quad \theta^* = \frac{\mu - r}{\sigma^2}.$$ 

This strategy is bounded and hence satisfies the required condition for admissibility. Moreover, if the coefficients of the model are such that $\mu > r$ and $\mu - r < \sigma^2$, this strategy takes values in the interval $(0, 1)$.

4.2 Stochastic control method

The stochastic control method is adequate for Markov models. This method consists in deriving the so called Hamilton-Jacobi-Bellman equation which allows the analysis of the model. Merton [56] was the first who applied stochastic control method to
a dynamic optimization problem in a standard Black-Scholes model with constant coefficients.

At time $t$ the portfolio manager is assumed to consume its wealth at consumption rate $c_t$, where by definition a consumption rate is a non-negative $\mathcal{F}_t$-adapted process $c = (c_t)_{t \in [0, T]}$ such that

$$\int_0^T c_t dt < \infty.$$  

With the additional consumption rate, the wealth process is formulated as controlled stochastic process satisfying

$$dX_t^{\theta, c} = [r X_t^{\theta, c} - c_t] dt + X_t^{\theta, c} \theta [\mu - r] dt + \sigma dW_t, \quad X_0^{\theta, c} = x,$$

and the dynamic optimization problem reads as follows

$$\text{find a pair } (\theta^*, c^*) \text{ which maximizes } E \left[ \int_0^T e^{-\alpha t} U_1(c_t) dt + U_2(X_T^{\theta, c}) \right],$$

for a given $\alpha \in (0, 1)$ and a utility functions $U_1$ and $U_2$. Let us define the value function of the maximization problem as

$$V(t, x) = \sup_{(\theta, c) \in (0, T]} E \left[ \int_t^T e^{-\alpha s} U_1(c_s) ds + U_2(X_T^{\theta, c}) \right].$$

The value function expresses the evaluation of the maximal value of the portfolio manager’s costs as function of the initial capital $x$ at the starting time $t$. Moreover, Bellman principle implies that

$$V(t, x) = \sup_{(\theta, c) \in (0, T]} E \left[ \int_t^u e^{-\alpha s} U_1(c_s) ds + V(u, X_u^{\theta, c}) \right],$$

for $u \leq T$. Itô’s lemma applied to the function $V(t, x)$ leads to the so called Hamilton-Jacobi-Bellman equation

$$\sup_{\alpha_1 \leq \theta \leq \alpha_2, c \geq 0} \left\{ \frac{1}{2} (x \sigma \theta)^2 V_{xx}(t, x) + (1 - \theta) x r + \theta x \mu - c \right\} V_x(t, x)$$

$$+ V_t(t, x) - r V(t, x) + U_1(c) \right\} = 0,$$

where $V(T, x) = U_2(x)$, $V(t, 0) = U_2(0)$,

where $\alpha_1, \alpha_1$ are real numbers. Solving this HJB equation leads to the optimal value function $V(t, x)$ as a solution of a partial differential equation obtained by substituting the optimal strategy and the consumption processes in the HJB-equation. These optimal processes are obtained here independently of the derivatives $V_t, V_x$ and $V_{xx}$.
4.3 Martingale method

In this section we address the portfolio optimization problem of a portfolio manager who wishes to maximize the expected utility from terminal wealth without any other restrictions such as risk management. This problem is solved by [Karatzas et al. [38], Cox and Huang [11]] using the so called martingale approach which we present in this section. As in Section 2, we consider a risk-free money market account with a constant interest rate \( r \) so that its price at time \( t \) is \( S^0_t = e^{-rt} \). Further we consider \( n \) risky stocks whose prices \( S_t = (S^1_t, \ldots, S^n_t) \) evolve according to

\[
dS^i_t = S^i_t [\mu^i dt + \sum_{j=1}^{n} \sigma_{ij} dW^j_t], \quad S^i_0 = s_0 \in \mathbb{R}, \quad i = 1, \ldots, n, \tag{4.6}
\]

with stock instantaneous mean returns \( \mu = (\mu_1, \ldots, \mu_n)^\top \) and the volatility matrix \( \sigma = (\sigma_{ij})_{i,j=1,\ldots,n} \) are assumed to be constants. Moreover, \( \sigma \) is assumed to satisfy the non-degeneracy condition

\[
x^\top \sigma \sigma^\top x \geq \delta x^\top x \quad \text{for all} \quad x \in \mathbb{R}^n,
\]

where \( \delta > 0 \) is a given constant.

The dynamic market completeness implies the existence of a unique state price density process \( H = (H_t)_{t \in [0,T]} \), given by

\[
dH_t = -H_t (rdt + \kappa^\top dW_t), \quad H_0 = 1, \tag{4.7}
\]

where \( \kappa := \sigma^{-1}(\mu - r1_n) \) is the market price of risk in the economy and \( 1_n \) is the \( n \)-dimensional vector whose all components are one.

The market price of risk \( \kappa \) or equivalently the state price density process \( H \) can be regarded as the driving economic parameter in a portfolio managers dynamic investment problem.

As it is outlined in Section 2, the portfolio manager is acting in this market using the portfolio process \( \theta = (\theta_t)_{t \in [0,T]}, \theta_t = (\theta^1_t, \ldots, \theta^n_t)^\top \) defined as the fractions of wealth invested in each risky stock, i.e.,

\[
\theta^i_t = \frac{\psi^i_t S^i_t}{X^\theta_0}, \quad i = 1, \ldots, n,
\]

with \( \theta^0_t = 1 - \sum_{i=1}^n \theta^i_t \) is the fraction of wealth invested in the risk-free bond. As a consequence, the wealth process can be formulated in terms of the portfolio process as a linear stochastic differential equation given by

\[
dx^\theta_t = X^\theta_t [r + \theta^\top_t (\mu - r1_n)] dt + X^\theta_t \theta^\top_t \sigma dW_t, \quad X^\theta_0 = x. \tag{4.8}
\]

At time \( t = T \) the portfolio manager reaches the terminal wealth \( X^\theta_T \). Thereby, the portfolio process \( \theta \) is assumed to be admissible in the following sense.
Definition 4.1.
Given \( x > 0 \), we say that a portfolio process \( \theta \) is admissible at \( x \), if the wealth process \( X^\theta = (X^\theta_t)_{t \in [0,T]} \) starting at \( X^\theta_0 = x \) satisfies \( P(X^\theta_t > 0, \text{ for all } t \in [0,T]) = 1 \). We shall denote by \( \mathcal{A}(x) \) the set of all admissible strategies.

Definition 4.2.
We call \( U : [0,\infty) \mapsto \mathbb{R} \cup \{-\infty\} \), a utility function if \( U \) is strictly increasing, strictly concave, twice continuously differentiable on \((0,\infty)\), and its derivative satisfies \( \lim_{x \to 0^+} U'(x) = \infty \) and \( \lim_{x \to \infty} U'(x) = 0 \). Further we denote by \( I : (0,\infty) \mapsto (0,\infty) \) the inverse function of \( U' \).

In this economy, the portfolio manager is assumed to derive from the terminal wealth \( X^\theta_T \) a utility \( U(X^\theta_T) \) and his objective is to maximize the expected utility by choosing an optimal strategy \( \theta^* \) from the set of admissible strategies \( \mathcal{A}(x) \).

The dynamic problem
Find a strategy \( \theta^* \) in \( \mathcal{A}(x) \) that solves

\[
\max_{\theta \in \mathcal{A}(x)} \mathbb{E}[U(X^\theta_T)].
\]  (4.9)

With no additional restrictions such as risk management, the maximization problem (4.9) was solved in the case of a complete market, by Cox and Huang [11] and independently by Karatzas, Lehocky and Shreve [38] using martingale and duality approaches.

In fact, Itô’s Formula implies that the process \((H_tX^\theta_t)_{t \in [0,T]}\) is a supermartingale which implies that the so called budget constraint

\[
\mathbb{E}[H_TX^\theta_T] \leq x
\]  (4.10)
is satisfied for every \( \theta \in \mathcal{A}(x) \). This means that the expected discounted terminal wealth can not exceed the initial wealth.

In the present case of a complete market, the following theorem is a basic tool in martingale method, see [41].

Theorem 4.3.
Let \( x > 0 \) be given and let \( \xi \) be a nonnegative, \( \mathcal{F}_T \)-measurable random variable such that

\[
\mathbb{E}[H_T \xi] = x.
\]

Then there exists a portfolio process \( \theta \) in \( \mathcal{A}(x) \) such that \( \xi = X^\theta_T \).

Define

\[
\mathcal{B}(x) := \{ \xi > 0 : \xi \text{ is } \mathcal{F}_T \text{-measurable and } \mathbb{E}[H_T \xi] \leq x \}.
\]

In contrast to the dynamic problem, where the portfolio manager is required to maximize the expected utility from terminal wealth over a set of processes, in a first
step the static problem is considered. Here the portfolio manager has the advantage to maximize only over a set of random variables which are the admissible terminal wealth.

**The static problem**
Find a random variable $\xi^*$ in $B(x)$ that solves

$$
\max_{\xi \in B(x)} \mathbb{E}[U(\xi)]. \tag{4.11}
$$

**Assumption**

$$
\mathbb{E}[H_T I(y H_T)] < \infty \quad \text{for all} \quad y \in (0, \infty).
$$

Under the last assumption the function $\chi : (0, \infty) \rightarrow (0, \infty)$ defined by $\chi(y) := \mathbb{E}[H_T I(y H_T)]$ inherits the property of being a continuous, strictly decreasing mapping of $(0, \infty)$ onto itself, and so $\chi$ has a (continuous, strictly decreasing) inverse $\chi^{-1}$ from $(0, \infty)$ onto itself. So we define

$$
\xi^* := I(\chi^{-1}(x) H_T),
$$

where $x$ is the initial endowment of the portfolio manager.

**Proposition 4.4.** The random variable $\xi^* := I(\chi^{-1}(x) H_T)$ satisfies

$$
\mathbb{E}[H_T \xi^*] = x, \tag{4.12}
$$

and for every portfolio $\theta \in \mathcal{A}(x)$, we have

$$
\mathbb{E}[U(X^\theta_T)] \leq \mathbb{E}[U(\xi^*)]. \tag{4.13}
$$

**Proof.** Equation (4.12) follows directly from the definition of $\xi^*$ and $\chi^{-1}$. The inequality (4.13) is a consequence of (4.12), (4.10) and of the following property of the utility function $U$ with the inverse of its derivative $I$, see [39]:

For $0 < x, y < \infty$, we have

$$
U(I(y)) \geq U(x) + y(I(y) - x).
$$

Theorem 4.5 which is proved in [41] solves the static optimization problem (4.11).

**Theorem 4.5.**

Consider the portfolio problem (4.9). Let $x > 0$ and set $y := \chi^{-1}(x)$, i.e., $y$ solves $x = E[H_T I(y H_T)]$. Then there exists for $\xi^* = \xi^N := I(y H_T)$, a self-financing portfolio process $\theta^N = (\theta^N_t)_{t \in [0,T]}$, such that

$$
\theta^N \in \mathcal{A}(x), \quad X^\theta^N_T = \xi^N,
$$

and the portfolio process solves the dynamic problem (4.9).
So far we have not used the assumption of market completeness, this assumption is used only to ensure the existence of the portfolio $\theta^N$ which finances $\xi^N$. However, when specialized to the case of logarithmic utility or to power utility defined in (3.5), one can directly use the Markov property of solutions of stochastic differential equations to express the current optimal wealth process $X^\theta_t$ before the horizon $T$, as a function of $H_t$ for which we apply Itô’s Formula. The optimal portfolio can be derived by equating coefficients with the wealth process given in (4.8), a task that we investigate in the following example.

Example 4.6. The problem of the so-called non-risk portfolio manager was studied by Cox and Huang [12], [11] where the manager has a power utility $U$ given by (3.5) with a constant relative risk aversion $\gamma$. According to Theorem 4.5, the static problem (4.11) has the optimal solution

$$
\xi^N = I(yH_T),
$$

with $I(x) = x^{-\frac{1}{\gamma}}$ is the inverse function of the derivative of the utility function $U$ and $y^N := \frac{1}{\gamma}e^{(1-\gamma)(r + \|\kappa\|^2/2)T}$.

Let $X^N_t$ be the optimal wealth before the horizon. Itô’s lemma applied to Equations (4.7) and (4.8) implies that the process $(H_tX^N_t)_{t\in[0,T]}$ is $\mathcal{F}$-martingale, i.e., $X^N_t = \mathbb{E}[\frac{H_T}{H_t}X^N_T|\mathcal{F}_t]$.

Here the optimal terminal wealth $X^N_T$ is given by Theorem 4.5 as $X^N_T := \xi^N = I(y^NH_T)$. Moreover, Markov’s property of the solution $(H_t)_{t\in[0,T]}$ of Equation (4.7) leads to the computation of this conditional expectation using the fact that $\ln H_t$ is normally distributed with mean $\ln H_t - (r + \|\kappa\|^2/2)(T - t)$ and variance $\|\kappa\|^2(T - t)$. We get for the optimal terminal wealth before the horizon the following form

$$
X^N_T = \frac{e^{\Gamma(t)}}{(y^NH_t)^{\frac{1}{\gamma}}} \quad \text{with} \quad \Gamma(t) := \frac{1-\gamma}{\gamma} \left( r + \frac{\|\kappa\|^2}{2\gamma} \right) (T - t).
$$

The optimal strategy is obtained by a representation approach. In this case, we have $X^N_t = F(H_t, t)$ with $F(z, t) = \frac{e^{\Gamma(t)}}{(y^Nz)^{\frac{1}{\gamma}}}$ for which we apply Itô’s lemma to get

$$
dX^N_t = \left[ F_t(H_t, t) - rF_z(H_t, t)H_t + \frac{\|\kappa\|^2}{2}F_{zz}(H_t, t)H_t^2 \right] dt - F_z(H_t, t)H_t\kappa^\top dW_t,
$$

where $F_z, F_{zz}$ and $F_t$ denote the partial derivatives of $F(z, t)$ w.r.t. $z$ and $t$, respectively. If we equate the volatility coefficient of this equation with the volatility coefficient of Equation (4.8), we derive the following constant optimal strategy

$$
\theta^N_t \equiv \theta^N = \frac{1}{\gamma}\sigma^{-1}\kappa = \frac{1}{\gamma}(\sigma\sigma^\top)^{-1}(\mu - r1_n) = \text{const}
$$

for the optimization in the absence of a risk-constraint which we call normal strategy.
4.4 The case of stochastic volatility

In [27] we consider an extended Merton’s problem of optimal consumption and investment in continuous-time with stochastic volatility. We approximate the linear wealth process by a particular weak Itô-Taylor approximation called Euler scheme, and then we show that the optimal control of the value function generated by the Euler scheme is an \( \varepsilon \)-optimal control of the original problem of maximizing total expected discounted utility from consumption. More precisely, we consider a financial market with a risk-free bond

\[
dS_t^0 = rS_t^0 dt,
\]

and one risky stock whose prices are given according to

\[
dS_t = S_t[\mu dt + \sigma(\Lambda_t) dW^1_t], \quad S = s_0 \in \mathbb{R}.
\]

The process \( \Lambda_t \) represents an economic factor with mean reversion property and can be observed by the investor. Models of stochastic volatility are of great interest in finance since they have the property to capture the stock return variability. Following Flemming and Hernandez [25], we assume that the dynamics of \( \Lambda_t \) is given by

\[
d\Lambda_t = g(\Lambda_t) dt + \beta[\rho dW^1_t + (1 - \rho^2)^{1/2} dW^2_t], \quad \Lambda_0 = \lambda \in \mathbb{R}
\]

with \( \mu > r \) and \(-1 \leq \rho \leq 1\). \( W^1 \) and \( W^2 \) are two independent Brownian motions. The functions \( \sigma \) and \( g \) are assumed to be in \( C^1(\mathbb{R}) \) and satisfy

- \( \sigma_z = \frac{d}{dx} \sigma \) is bounded and \( \sigma_l \leq \sigma(.) \leq \sigma_u \) for a given constants \( 0 < \sigma_l < \sigma_u \);
- \( g_z = \frac{d}{dx} g \) is bounded and there exists \( k > 0 \) such that \( g_z \leq -k \).

We denote by \( \theta_t \) the trading portfolio corresponding to the fraction of wealth invested in the risky asset at time \( t \), and let \( \tilde{\epsilon}_t = \frac{\theta_t}{X_t} \) be the fraction of wealth which is consumed. Then the wealth process satisfies

\[
dx_t^\theta = X_t^\theta [(r + (\mu - r)\theta_t - \tilde{\epsilon}_t) dt + \theta_t \sigma(\Lambda_t) dW^1_t], \quad X_0^\theta = x > 0,
\]

where the processes \( \theta = (\theta_t)_{t \in [0,T]} \) and \( \tilde{\epsilon} = (\tilde{\epsilon}_t)_{t \in [0,T]} \) are supposed to be admissible in the sense that they are \( \mathcal{F}_t \)-progressively measurable such that \( P[\forall t > 0 : |\theta_t| \leq A_1, 0 \leq \tilde{\epsilon}_t \leq A_2] = 1 \), where \( A_1 \) and \( A_2 \) are given constants. Let \( \mathcal{A}(x) \) denote the set of admissible strategies.

The portfolio manager’s objective is to find an investment portfolio and a consumption rate so as to maximize over the set of admissible strategies the following expected total discounted utility from consumption

\[
J(x, \Lambda; \theta, \tilde{\epsilon}) = \mathbb{E} \left[ \int_0^\infty \frac{1}{1 - \gamma} e^{-\alpha t}(\tilde{\epsilon}_t X_t^\theta)^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1.
\]
This is well defined since it is shown in [25] that the value function

\[ V(x, \Lambda) = \sup_{\theta, \tilde{c} \in A(x)} \mathbb{E} \left[ \int_0^\infty \frac{1}{1 - \gamma} e^{-\alpha t}(\tilde{c}_t X^\theta_t)^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1, \]

is bounded. This problem is solved by Flemming and Hernandez in [25], where the authors write \( V(x, \Lambda) \) in the form \( V(x, \Lambda) = \frac{\alpha}{\gamma} \tilde{V}(\Lambda) \) for some function \( \tilde{V} \), then, by a change of probability measure argument the portfolio optimization problem was reduced to a stochastic control problem for which the dynamic programming equation is a non linear differential equation with state \( \Lambda_t \) given by equation (4.16), and they show that the value function \( \tilde{V}(\Lambda) \) is the unique positive classical solution to the dynamic programming equation associated with this stochastic problem. In particular, they obtained via analytic and stochastic control arguments optimal investment and consumption policies as feedback policies of the current wealth \( X^\theta \) which we approximate by a particular stochastic Itô-Taylor approximation called Euler scheme. Let us restrict our study to the case of finite time horizon \( T > 0 \) and let

\[ V(x, \Lambda; T) = \sup_{\theta, \tilde{c} \in A(x)} J(x, \Lambda; \theta, \tilde{c}, T) \tag{4.19} \]

be the associated optimal costs function.

The Euler approximation of the wealth process (4.17) is defined for a given discretization \( 0 = t_0 < t_1 < \cdots < t_N = T \) of the time interval \([0, T]\) by

\[ Y^\theta_N(t_{i+1}) = Y^\theta_N(t_i) + (r + \theta_m(\mu - r) - \tilde{c}_{m_i})Y^\theta_N(t_i)(t_{i+1} - t_i) + \sigma(\Lambda(t_i))\theta_{m_i} Y^\theta_N(t_i)(W_{t_{i+1}} - W_{t_i}), \]

for \( i = 0, 1, 2, \ldots, N - 1 \) with initial value \( Y^\theta_0 = x \). We shall use the interpolated Euler scheme which is defined by

\[ Y^\theta_N(t) = Y^\theta_N(t_{m_i}) + (r + \theta_{m_i}(\mu - r) - \tilde{c}_{m_i})Y^\theta_N(t_{m_i})(t - t_{m_i}) + \sigma(\Lambda(t_{m_i}))\theta_{m_i} Y^\theta_N(t_{m_i})(W_t - W_{t_{m_i}}), \tag{4.20} \]

with \( n_i := \max\{0 \leq i \leq N, t_i \leq t\} \). This approximation is a particular case of the truncated Itô-Taylor expansion which has the important property that it allows a sufficiently smooth function of an Itô process to be expanded as the sum of a finite number of terms represented by multiple Itô integrals with constant integrands and a remainder consisting of a finite number of other multiple Itô integrals with non constant integrands. We refer to [45] for more details. In [27] we first prove that the functional

\[ J_N(x, \Lambda; \theta, \tilde{c}, T) = \mathbb{E} \left[ \int_0^T \frac{1}{1 - \gamma} e^{-\alpha t}(\tilde{c}_t Y^\theta_N(t))^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1 \tag{4.21} \]

is continuous with respect to the controls \( \theta, \tilde{c} \in A(x) \), i.e., if \( u^m = (\theta^m, c^m) \) is a sequence of admissible controls converging uniformly in \( L^\infty(\Omega) \) to a given admissible control \( u = (\theta, \tilde{c}) \), i.e.,

\[ \sup_{0 \leq t \leq T} \|\theta^m_t - \theta_t\|_{L^\infty(\Omega)}; \quad \sup_{0 \leq t \leq T} \|\tilde{c}^m_t - \tilde{c}_t\|_{L^\infty(\Omega)} \to 0 \text{ as } m \to \infty. \]
Then, we have
\[ |J_N(x; \Lambda; u^m, T) - J_N(x; \Lambda; u, T)| \longrightarrow 0 \text{ as } m \rightarrow \infty. \]

Further we have proved in [27] that the interpolated Euler approximation process defined in (4.20) converges strongly to the wealth process given by (4.17), i.e.,
\[ E[|Y^\theta_N(T) - X^\theta_N|^2] \longrightarrow 0 \text{ as } N \rightarrow \infty. \]

From the other hand, the conditions of (Theorem 14.5.1 [45]) are satisfied in our model so that we have the convergence of the Euler scheme in the weak sense and as consequence we prove in [27] the following main result.

**Theorem 4.7.**

For \( T > 0, \theta, c \in \mathcal{A}(x) \) let
\[ J(x; \Lambda; \theta, \tilde{c}, T) = E\left[ \int_0^T \frac{1}{1 - \gamma} e^{-\alpha t}(\tilde{c}_t X^\theta_t)^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1 \]
be the total discounted utility from consumption, where \( X^\theta \) is the wealth process given in (4.17). Let \( Y_N(t) \) be its interpolated Euler approximation and
\[ J_N(x; \Lambda; \theta, \tilde{c}, T) = E\left[ \int_0^T \frac{1}{1 - \gamma} e^{-\alpha t}(\tilde{c}_t Y^\theta_N(t))^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1 \]
its associated total discounted utility of consumption. Let \( u^*_t = (\theta^*_t, c^*_t) \) be an optimal control of optimal costs function
\[ V(x; \Lambda; T) = \sup_{\theta, \tilde{c} \in \mathcal{A}(x)} J(x; \Lambda; \theta, \tilde{c}, T), \quad (4.22) \]
and let \( u^*_N(t) = (\theta^*_N(t), \tilde{c}^*_N(t)) \) be an optimal control of the value function
\[ V_N(x; \Lambda; T) = \sup_{\theta, \tilde{c} \in \mathcal{A}(x)} J_N(x; \Lambda; \theta, \tilde{c}, T). \quad (4.23) \]

Then, we have
\[ \forall \epsilon > 0 \text{ there exists a partition } (t_i)_{0 \leq i \leq N}, \text{ such that} \]
\[ |J(x; \Lambda; u^*_t, T) - J(x; \Lambda; u^*_N(t), T)| < \epsilon \]
for every \( N \geq N_\epsilon. \)

**Proof.** Let \( T > 0, u^*_t = (\theta^*_t, \tilde{c}^*_t) \) be an optimal policy of the original problem
\[ V(x; \Lambda, T) = \sup_{\theta, \tilde{c} \in \mathcal{A}(x)} E\left[ \int_0^T \frac{1}{1 - \gamma} e^{-\alpha t}(\tilde{c}_t X^\theta_t)^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1. \]
We denote by \( u_N^*(t) = (\theta_N^*(t), \varphi_N^*(t)) \) the optimal control of the objective function
\[
V_N(x, \Lambda) = \sup_{\theta, z \in A(x)} \mathbb{E} \left[ \int_0^T \frac{1}{1 - \gamma} e^{-\alpha t} \tilde{c}_i Y_N^\theta(t) \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1.
\]
associated with the controlled interpolated Euler scheme \( Y_N^\theta \).

Then, we have
\[
J(x, \Lambda; u_i^*, T) - J(x, \Lambda; u_N^*(t), T) = J(x, \Lambda; u_i^*, T) - J_N(x, \Lambda; u_i^*, T) + J_N(x, \Lambda; u_i^*, T) - J_N(x, \Lambda; u_N^*(t), T) + J_N(x, \Lambda; u_N^*(t), T) - J(x, \Lambda; u_N^*(t), T).
\]

Since \( u_N^*(t) \) is the global optimal control associated with \( J_N \), it follows that
\[
J_N(x, \Lambda; u_i^*, T) - J_N(x, \Lambda; u_N^*(t), T) \leq 0.
\]

From the one hand, we have
\[
|J(x, \Lambda; u_i^*, T) - J_N(x, \Lambda; u_i^*, T)| = \mathbb{E} \left[ \int_0^T e^{-\alpha T} \tilde{c}_i (X_i^\theta)^{1-\gamma} - (Y_N^\theta(t))^{1-\gamma} dt \right] \leq \frac{A_1^{1-\gamma}}{1 - \gamma} \int_0^T e^{-\alpha T} \mathbb{E} \left[ g(X_i^\theta) - g(Y_N^\theta(t)) \right] dt,
\]
with \( g(x) = x^{1-\gamma} \) for \( x > 0 \). The function \( g \) as well the coefficient \( a(t, x) = x[r + \pi_t(b - r) - \tilde{c}_i], b(t, x) = \sigma(z) \pi_t x \) of the wealth process satisfy the conditions of (Theorem 14.5.1 [45]), and therefore there exists a positive constant \( C_g \) such that
\[
|J(x, \Lambda; u_i^*, T) - J_N(x, \Lambda; u_i^*, T)| \leq \frac{A_1^{1-\gamma}}{1 - \gamma} \left( \int_0^T e^{-\alpha T} dt \right) C_g \sup_{1 \leq i \leq N} |t_i - t_{i-1}|
\]
and
\[
|J_N(x, \Lambda; u_N^*(t), T) - J(x, \Lambda; u_N^*(t), T)| \leq \frac{A_2^{1-\gamma}}{1 - \gamma} \left( \int_0^T e^{-\alpha T} dt \right) C_g \sup_{1 \leq i \leq N} |t_i - t_{i-1}|.
\]

On the other hand, the right hand side of (4.24) is positive since \( u^* \) is global optimal control, it follows that
\[
|J(x, \Lambda; u_i^*, T) - J(x, \Lambda; u_N^*(t), T)| \leq 2 \frac{A_2^{1-\gamma}}{1 - \gamma} \left( \int_0^T e^{-\alpha T} dt \right) C_g \sup_{1 \leq i \leq N} |t_i - t_{i-1}|.
\]

**Remark 4.8.** We emphasize the important fact that the main goal of a stochastic time-discrete approximation is a practical simulation of solutions of stochastic differential equations needed in situations where a good pathwise approximation is required, or in situations dealing with expectations of functionals of an Itô process which can not be determined analytically. From this fact the last result could be interesting for numerical computations.
5 Risk measures

In general, a risk is related to the possibility of losing wealth and assumed to be an undesirable characteristic of a random outcome of a given financial investment. Without other considerations such as risk constraints, the optimal portfolio strategy given in Example (4.6) leads (by definition) to the maximum expected utility of the terminal wealth. Nevertheless, these strategies are more risky since they lead to extreme positions, and as consequence the optimal terminal wealth does not exceed the initial investment with a high probability. This distribution is not desirable for a portfolio manager, who offers, say, a life insurance with a fixed minimum rate of return. In order to incorporate such shortfall risks into the optimization it is necessary to quantify them by using appropriate risk measures. By the term shortfall risk we denote the event, that the terminal wealth falls below some threshold value $Q > 0$.

In Section 8 we shall introduce a shortfall level which is related to the result of an investment into the money market. A typical choice is

$$Q := q = e^{\delta T} X_T^{\theta \equiv 0} = xe^{(r+\delta)T}, \quad \delta \in \mathbb{R}.$$  

Here, $X_T^{\theta \equiv 0} = xe^{rT}$ is the terminal wealth of a pure bond portfolio where the portfolio manager follows the buy-and-hold strategy $\theta \equiv 0$. In this case shortfall means to reach not an target interest rate of $r+\delta$. For $\delta = -r$ we have $q = x$, i.e., the shortfall level is equal to the initial capital.

In Section 11 we deal with a stochastic shortfall level $Q$, which is proportional to the result of an investment into a pure stock portfolio managed by the buy-and-hold strategy $\theta \equiv 1$. Especially we set for $S_0 = 1$

$$Q = e^{\delta T} X_T^{\theta \equiv 1} = e^{\delta T} xS_T, \quad \delta \in \mathbb{R}.$$  

The shortfall risk consists of the random event $C = \{X_T < Q\}$ or $\{G := X_T - Q < 0\}$. Next we assign risk measures to the random variable (risk) $G$ and denote them by $\rho(G)$. Using these measures, constraints of the type $\rho(G) \leq \varepsilon$ for some $\varepsilon > 0$ can be added to the formulation of the portfolio optimization problem.

We now present some risk measures used in this thesis. A natural idea is to restrict the probability of a shortfall, i.e.,

$$\rho_1(G) = P(G < 0) = P(X_T < Q) \leq \varepsilon.$$  

Here $\varepsilon \in (0, 1)$ is the maximum shortfall probability which is accepted by the portfolio manager. This approach corresponds to the widely used concept of Value at Risk (VaR) which is defined as

$$\text{VaR}_\varepsilon(G) = -\zeta_\varepsilon(G),$$  

where $\zeta_\varepsilon(G)$ denotes the $\varepsilon$-quantile of the random variable $G$. VaR can be interpreted as the threshold value for the risk $G$ such that $G$ falls short this value with some given probability $\varepsilon$. It holds

$$P(G < 0) \leq \varepsilon \iff \text{VaR}_\varepsilon(G) \leq 0 \iff \text{VaR}_\varepsilon(X_T) \leq -Q.$$
Value at Risk is the most common tool in risk management for banks and many financial institutions. It is defined as the worst loss for a given confidence level. For a confidence level of $\alpha = 99\%$ one is $99\%$ certain that at the end of a chosen risk horizon there will be no smaller wealth than the VaR. In the academic literature many works have focused on the Value at Risk as risk measure, see for example Duffie and Pan [19]. Theoretical properties of the Value at risk are discussed in Artzner [2], Cvitanic and Karatzas [13].

As it is pointed out in [28], the VaR risk measure has the shortcoming to control only the probability of loss rather than its magnitude and as consequence the expected losses in the states where there are large losses are higher than the expected losses the portfolio manager would have incurred by avoiding the use of VaR risk measure. In order to overcome this shortcoming of the VaR, the risk manager uses as alternative the so-called Expected Loss denoted by EL and defined as

$$
\rho_2(G) = \text{EL}(G) := E[G^-] = E[(X_T - Q)^-].
$$

Since the aim is to maximize the expected utility of the terminal wealth $X_T$, one can also compare the utilities of $X_T$ and of a given benchmark $Q$.

Let $U$ denote a utility function given by Definition 4.2. Realizations of $X_T$ with $U(X_T)$ below the target utility $U(Q)$ are those of an unacceptable shortfall. Then the random event $C$ can also be written as $C = \{X_T < Q\} = \{U(X_T) < U(Q)\}$. Defining the random variable $G = G(X_T, Q) = U(X_T) - U(Q)$ we have $C = \{G < 0\}$. The random variable $G$ can be interpreted as the utility gain of the terminal wealth relative to the benchmark. In order to quantify the shortfall risk we assign to the random variable $G$ a real-valued risk measure $\rho(G)$ given by

$$
\rho_3(G) = \text{EUL}(G) := E[G^-] = E[(U(X_T) - U(Q))^-],
$$

and call it Expected Utility Loss (EUL). Here similarly to the EL risk measure, the risk measure (VaR) can be defined in terms of the utility function in the following sense

$$
\rho_1(G) = P(G < 0) = P(U(X_T) < U(Q)) = P(X_T < Q)
$$

since $U$ is strictly increasing. Further risk measures can be found in the class of coherent measures introduced by Artzner, Delbaen, Eber and Heath [13] and Delbaen [14] where the characteristics of a risk function $\rho(X)$ have been proposed.

**Definition 5.1.** Consider a set $V$ of real-valued random variables. A function $\rho : V \rightarrow \mathbb{R}$ is called coherent risk measure if it is

(i) *monotonous:* $X \in V, X \geq 0 \implies \rho(X) \geq 0$

(ii) *sub-additive:* $X, Y, X + Y \in V \implies \rho(X + Y) \leq \rho(X) + \rho(Y)$

(iii) *positively homogeneous* $X \in V, h > 0, X \in V \implies \rho(hX) = h\rho(X)$

(iv) *translation invariant* $X \in V, a \in R \implies \rho(X + a) = \rho(X) - a.$
Delbaen [14] proved that the VaR measure is not a coherent risk measure since it
does not fulfil the sub-additivity property. This property expresses the fact that a
portfolio made of sub-portfolios will risk an amount which is at most the sum of the
separate amounts risked by its sub-portfolios.
EUL and EL risk measures do not belong to the class of coherent risk measures, since
they both violate the translation property. We refer to Basak, Shapiro et.al. [4, 5] and
our papers [28, 31], where VaR-based risk measures are used as constraints of portfolio
optimization problems. Constraints modeling the Expected Loss and the Expected
Utility Loss are studied within a portfolio maximization problem in [28, 30, 31].
We shall discuss in the next chapters the behavior of a portfolio manager who wants
to maximize its expected utility from terminal wealth in presence of different shortfall
risks measured by the last discussed risk measures.

Remark 5.2. If we choose a logarithmic utility function \( U(z) = \ln z \), i.e., we set
\( \gamma = 1 \), then the maximization of the expected utility \( \mathbb{E}[U(X^\theta_T)] \) of the terminal wealth
of the portfolio \( \theta \), is equivalent to the maximization of the expected annual logarithmic
return of this portfolio. The annual logarithmic return is defined as

\[
L(X^\theta_T) := \frac{1}{T} \ln \frac{X^\theta_T}{x}
\]

where \( x \) is the initial capital, i.e., \( X_0 = x \). Hence, we find

\[
\mathbb{E}[L(X^\theta_T)] := \frac{1}{T} \mathbb{E}[U(X^\theta_T) - U(x)].
\]

For the Expected Utility Loss we derive

\[
\mathbb{E}[(U(X^\theta_T) - U(Q))^+] = T \mathbb{E}[(L(X^\theta_T) - L(Q))^+].
\]

It can be seen, that bounding the Expected Utility Loss by \( \varepsilon \) is equivalent to bounding
the Expected Loss of the annual logarithmic return by \( \frac{\varepsilon}{T} \).