Chapter 4

A Generalized Cahn-Hilliard Equation based on a Microforce Balance

4.1 Derivation of the Model

We start again with the derivation of the classical Cahn-Hilliard equation. Consider the free energy functional of the form

\[ F(\psi) = \int_{\Omega} \left( \frac{1}{2} |\nabla \psi|^2 + \Phi(\psi) \right) \, dx, \tag{4.1} \]

where \( \Omega \) is a bounded, open and connected subset of \( \mathbb{R}^n \) with boundary \( \Gamma := \partial \Omega \in C^3 \). We assume that the order parameter \( \psi \) is a conserved quantity. The according conservation law reads

\[ \partial_t \psi + \text{div} \, j = 0, \tag{4.2} \]

where \( j \) is a vector field representing the phase flux of the order parameter. The next step is to combine the two quantities \( j \) and \( \mu \). Similar to Fourier’s law in the derivation of the heat equation one typically assumes that \( j \) is given by

\[ j = -\nabla \mu, \tag{4.3} \]

a postulated relation. Finally we have to derive an equation for \( \mu \). The chemical potential \( \mu \) is given by the variational derivative of \( F \), i.e.

\[ \mu = \frac{\delta F}{\delta \psi} = -\Delta \psi + \Phi'(\psi). \]

If \( F \) is of the form (4.1) this yields the classical Cahn-Hilliard equation.

In the early nineties Gurtin [16] proposed a generalized Cahn-Hilliard equation, which is based on the following objections:

- Fundamental physical laws should account for the work associated with each operative kinematical process;
- There is no clear separation of the balance law (4.2) and the constitutive equation (4.3);
- Forces that are associated with microscopic configurations of atoms are not considered in the derivation of the classical Cahn-Hilliard equation.

According to Gurtin there should exist so called 'microforces' whose work accompanies changes in the order parameter \( \psi \). The microforce system is characterized by the microstress \( \xi \in \mathbb{R}^n \) and
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scalar quantities \( \pi \) and \( \gamma \) which represent internal and external microforces, respectively. The main assumption in [16] is that \( \xi, \pi \) and \( \gamma \) satisfy the (local) microforce balance

\[
\text{div} \, \xi + \pi + \gamma = 0, \tag{4.4}
\]

which can be motivated from a static point of view, see [16] for more details. In a next step we want to derive constitutive equations, which relate the quantities \( j \), the flux of the order parameter, \( \xi \) and \( \pi \) to the fields \( \psi \) and \( \mu \). The technique used in [16] for this derivation is based on the balance equation (4.4) and a (local) dissipation inequality, which is a direct consequence of the first and the second law of thermodynamics, that is, the energy balance

\[
\frac{d}{dt} \int_\Omega e \, dx = - \int_{\partial \Omega} q \cdot \nu \ d\sigma + \int_\Omega r \, dx + W(\Omega) + \mathcal{M}(\Omega),
\]

and

\[
\frac{d}{dt} \int_\Omega S \, dx \geq - \int_{\partial \Omega} \frac{q}{\theta} \cdot \nu \ d\sigma + \int_\Omega \frac{r}{\theta} \, dx,
\]

cf. [16, Appendix A]. The second law of thermodynamics is also known as the Clausius-Duhem inequality. Here \( e \) is the internal energy, \( S \) is the entropy, \( \theta \) is the absolute temperature, \( q \) is the heat flux, \( r \) is the heat supply, \( W(\Omega) \) is the rate of working on \( \Omega \) of all forces exterior to \( \Omega \) and \( \mathcal{M}(\Omega) \) is the rate at which energy is added to \( \Omega \) by mass transport. Let \( F \) be the free energy density, depending on the vector \( z = (\psi, \nabla \psi, \mu, \nabla \mu, \partial_t \psi) \). Then the second law of thermodynamics (in its mechanical version as considered by Gurtin [16]) reads

\[
\frac{d}{dt} \int_\Omega F(z) \, dx \leq - \int_{\partial \Omega} \mu j(z) \cdot \nu \ d\sigma + \int_\Omega \xi \cdot \nabla \partial_t \psi \ d\sigma + \int_\Omega \mu m \, dx + \int_\Omega \gamma \partial_t \psi \, dx,
\]

with \( m \) being the external mass supply. Making use of Green’s formula, we obtain

\[
\frac{d}{dt} \int_\Omega F(z) \, dx \leq - \int_\Omega \left( \nabla \cdot j(z) + \mu \text{div} \, j \right) \, dx
\]

\[
\quad + \int_\Omega \left( \text{div} \, \partial_t \psi + \xi \cdot \nabla \partial_t \psi \right) \, dx + \int_\Omega \mu m \, dx + \int_\Omega \gamma \partial_t \psi \, dx.
\]

In presence of external mass supply \( m \), (4.2) will be modified to

\[
\partial_t \psi + \text{div} \, j = m. \tag{4.5}
\]

In view of (4.4) and (4.5) we obtain the dissipation inequality

\[
\frac{d}{dt} \int_\Omega F(z) \, dx \leq \int_\Omega \left( \mu \partial_t \psi - j \cdot \nabla \mu - \pi \partial_t \psi + \xi \cdot \nabla \partial_t \psi \right) \, dx.
\]

This in turn yields the following local dissipation inequality

\[
\partial_t F(z) \leq \mu \partial_t \psi - j \cdot \nabla \mu - \pi \partial_t \psi + \xi \cdot \nabla \partial_t \psi,
\]

for all fields \( \psi \) and \( \mu \), this means, we have

\[
(\partial_\psi F + \pi - \mu) \dot{\psi} + (\partial_\psi \varphi F - \xi) \cdot \nabla \psi + \partial_\mu F \dot{\mu} + \partial_\psi F \nabla \dot{\mu} + \partial_\psi F \dot{\psi} + \nabla \mu \cdot j \leq 0, \tag{4.6}
\]

where \( \dot{u} = \partial_t u \) and \( \ddot{u} = \partial^2_t u \) for a smooth function \( u \). This local inequality needs to be satisfied for all smooth fields \( \psi \) and \( \mu \). Hence we have necessarily

\[
F(z) = F(\psi, \nabla \psi) \quad \text{and} \quad \xi(\psi, \nabla \psi) = \partial_\psi \varphi F(\psi, \nabla \psi)
\]

and there remains the inequality

\[
(\partial_\psi F + \pi - \mu) \dot{\psi} + \nabla \mu \cdot j \leq 0
\]
whose general solution is given by (cf. [16, Appendix B])

\[ \partial_t F + \pi - \mu = -\beta \dot{\psi} - c \cdot \nabla \mu \quad \text{and} \quad j = -a \dot{\psi} - B \nabla \mu, \]

with constitutive moduli \( \beta(z) \) (scalar), \( a(z) \), \( c(z) \) (vectors), \( B(z) \) (matrix) and the constraint that the tensor

\[
\begin{bmatrix}
\beta & c^T \\
a & B
\end{bmatrix}
\]

is positive semidefinite. We assume that \( \beta \) is constant and \( a, c \) and \( B \) do only depend on \( x \) instead of \( z \), whence we deal with an approximation of the constitutive moduli \( \beta(z), a(z), B(z) \). In particular, if the free energy density \( F \) is given by \( F(\psi, \nabla \psi) = \frac{1}{2} |\nabla \psi|^2 + \Phi(\psi) \) we obtain the following Cahn-Hilliard-Gurtin equations.

\[
\begin{align*}
\partial_t \psi - \text{div}(B \nabla \mu) - \text{div}(a \partial_t \psi) &= f, & t \in J, \ x \in \Omega, \\
\mu - c \cdot \nabla \mu + \Delta \psi - \beta \partial_t \psi - \Phi'(\psi) &= g, & t \in J, \ x \in \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n \) is open, bounded with compact boundary \( \Gamma = \partial \Omega \subset C^3 \). In this chapter, we are interested in solutions of (4.8) subject to the Neumann boundary conditions \( \partial_n \psi = 0 \) and \( B \nabla \mu \cdot \nu = 0 \), having optimal regularity in the sense

\[
\psi \in H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^2_p(\Omega)),
\]

and

\[
\mu \in L_p(J; H^2_p(\Omega)).
\]

We impose the following assumptions on the data \( a, c \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^n) \) and \( B \in C^1_{ub}(\overline{\Omega}; \mathbb{R}^{n \times n}) \).

\[
\text{div} a(x) = \text{div} c(x) = 0, \quad \text{for all} \ x \in \Omega, \quad \text{and} \quad a(x) \cdot \nu(x) = c(x) \cdot \nu(x) = 0, \quad \text{for all} \ x \in \Gamma,
\]

\[
B(x) \tau(x) \cdot \nu(x) = 0, \quad \text{for all} \ x \in \Gamma \text{ and all} \ \tau(x) \in T_x \Gamma,
\]

where \( T_x \Gamma \) denotes the tangential space in a point \( x \in \Gamma \) on \( \Gamma \).

Finally we want to emphasize that for the special case \( B = I, \ a = c = 0 \) and \( \beta = 0 \), we obtain the classical Cahn-Hilliard equation.

### 4.2 The Linear Cahn-Hilliard-Gurtin Problem in \( \mathbb{R}^n \)

In this section we will solve the full space problem

\[
\begin{align*}
\partial_t u - \text{div}(a \partial_t u) &= \text{div}(B \nabla \mu) + f, & t > 0, \ x \in \mathbb{R}^n, \\
\mu - c \cdot \nabla \mu &= \beta \partial_t u - \Delta u + g, & t > 0, \ x \in \mathbb{R}^n,
\end{align*}
\]

\[
u(0) = u_0, \quad t = 0, \ x \in \mathbb{R}^n,
\]

where \( \beta \in \mathbb{R}_+, \ a, c \in \mathbb{R}^n \) and \( B \in \mathbb{R}^{n \times n} \) is symmetric and positive definite. Set \( A = \beta B - \frac{1}{2}(a \otimes c + c \otimes a) \), where \( a \otimes c = (a_i c_j)_{i,j=1}^n \). In the sequel we assume the following condition on the matrix \( A \).

(A) There is a constant \( \varepsilon > 0 \), such that \( (A \xi | \xi) \geq \varepsilon |\xi|^2 \) for all \( \xi \in \mathbb{R}^n \).

Here is the main result on optimal \( L_p \)-regularity of (4.11).

**Theorem 4.2.1.** Let \( 1 < p < \infty \) and assume that (A) holds true. Then (4.11) admits a unique solution

\[
\begin{align*}
u \in H^1_p(J; H^1_p(\mathbb{R}^n)) \cap L_p(J; H^2_p(\mathbb{R}^n)) &=: Z^1, \\
\mu \in L_p(J; H^2_p(\mathbb{R}^n)) &=: Z^2,
\end{align*}
\]

if the data is subject to the following conditions.
Then it holds that $w, \eta$

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(i) $f \in L_p(\mathcal{J}; L_p(\mathbb{R}^n)) =: X^1,$

(ii) $g \in L_p(\mathcal{J}; H^1_p(\mathbb{R}^n)) =: X^2,$

(iii) $u_0 \in B^{4-2/p}_{pp}(\mathbb{R}^n) =: X_p.$

Proof. We apply the operator $(I - \Delta)^{-1/2}$ to both equations in (4.11) and define the new functions $w = (I - \Delta)^{-1/2} u$, $\eta = (I - \Delta)^{-1/2} \mu$, $\tilde{f} = (I - \Delta)^{-1/2} f$, $\tilde{g} = (I - \Delta)^{-1/2} g$ and $w_0 = (I - \Delta)^{-1/2} u_0$. Then it holds that

$$\tilde{f} \in L_p(\mathcal{J}; H^1_p(\mathbb{R}^n)), \quad \tilde{g} \in L_p(\mathcal{J}; H^2_p(\mathbb{R}^n)),$$

$$w_0 \in B^{4-2/p}_{pp}(\mathbb{R}^n)$$

and we are looking for a solution $(w, \eta)$ of the system

$$w_t - \text{div}(aw_t) = \text{div}(B\nabla \eta) + \tilde{f}, \quad t > 0, \ x \in \mathbb{R}^n,$$

$$\eta - c \cdot \nabla \eta = \beta w_t - \Delta w + \tilde{g}, \quad t > 0, \ x \in \mathbb{R}^n,$$

$$w(0) = w_0, \quad t = 0, \ x \in \mathbb{R}^n,$$

in the regularity class

$$w \in H^1_p(\mathcal{J}; H^2_p(\mathbb{R}^n)) \cap L_p(\mathcal{J}; H^4_p(\mathbb{R}^n)),$$

$$\eta \in L_p(\mathcal{J}; H^3_p(\mathbb{R}^n)).$$

In a next step we want to eliminate the functions $\tilde{g}$ and $w_0$. To achieve this, let $w^*$ be the unique solution of the problem

$$\beta w^*_t - \Delta w^* = -\tilde{g}, \quad t > 0, \ x \in \mathbb{R}^n,$$

$$w^*(0) = w_0, \quad t = 0, \ x \in \mathbb{R}^n,$$

with regularity

$$w^* \in H^1_p(\mathcal{J}; L_p(\mathbb{R}^n)) \cap L_p(\mathcal{J}; H^2_p(\mathbb{R}^n)),$$

if and only if $\tilde{g} \in L_p(J \times \mathbb{R}^n)$ and $w_0 \in B^{2-2/p}_{pp}(\mathbb{R}^n)$. Here $J$ denotes the interval $[0, T]$. If we even have $\tilde{g} \in L_p(\mathcal{J}; H^2_p(\mathbb{R}^n))$ and $w_0 \in B^{4-2/p}_{pp}(\mathbb{R}^n)$ then by regularity theory we obtain

$$w^* \in H^1_p(\mathcal{J}; H^2_p(\mathbb{R}^n)) \cap L_p(\mathcal{J}; H^4_p(\mathbb{R}^n)).$$

The pair of functions $(v, \eta) = (w - w^*, \eta)$ should now solve the problem

$$\partial_t v - \text{div}(av_t) = \text{div}(B\nabla \eta) + F, \quad t > 0, \ x \in \mathbb{R}^n,$$

$$\eta - c \cdot \nabla \eta = \beta \partial_t v - \Delta v, \quad t > 0, \ x \in \mathbb{R}^n,$$

$$v(0) = 0, \quad t = 0, \ x \in \mathbb{R}^n,$$

where $F$ is defined by

$$F = \tilde{f} + w^*_t - \text{div}(aw^*_t) \in L_p(\mathcal{J}; H^1_p(\mathbb{R}^n)).$$

In order to solve (4.13) we take the Laplace transform in the time variable and the Fourier transform in the spatial variable to obtain

$$\lambda (1 - i(a|\xi)) \hat{v} = -(B\xi|\xi) \hat{\eta} + \hat{F},$$

$$(1 - i(c|\xi)) \hat{\eta} = (\beta \lambda + |\xi|^2) \hat{v},$$

and $(\cdot | \cdot)$ denotes the inner product in $\mathbb{C}^n$. This system of algebraic equations can be written in matrix form

$$\begin{bmatrix}
\lambda (1 - i(a|\xi)) \\
-(\beta \lambda + |\xi|^2)
\end{bmatrix} \begin{bmatrix}
\hat{v} \\
\hat{\eta}
\end{bmatrix} = \begin{bmatrix}
\hat{F} \\
0
\end{bmatrix},$$

$$M(\lambda, \xi) \begin{bmatrix}
\hat{v} \\
\hat{\eta}
\end{bmatrix} = \begin{bmatrix}
\hat{F} \\
0
\end{bmatrix},$$

$$M(\lambda, \xi) = \begin{bmatrix}
\lambda (1 - i(a|\xi)) & (B\xi|\xi) \\
-(\beta \lambda + |\xi|^2) & (1 - i(c|\xi))
\end{bmatrix}.$$
Show that the symbol $\hat{\alpha}$ where

Therefore it holds that

Let $v_0, v_1 \in \mathcal{A}_p^n(J; H^2_p(\mathbb{R}^n))$ be the unique solutions of

and

Therefore it holds that

and

where $T$ is defined by its Fourier-Laplace symbol

The assertion of the theorem follows if we can show that $T$ is a bounded operator from $L_p(J; L_p(\mathbb{R}^n))$ to $L_p(J; L_p(\mathbb{R}^n))$. This will be a consequence of the classical Mikhlin multiplier theorem and the Kalton-Weis Theorem 1.3.1. We recall the classical Mikhlin condition

where $\alpha \in \mathbb{N}_0^n$ is a multiindex and $[s]$ denotes the largest integer not exceeding $s \in \mathbb{R}$. Firstly we show that the symbol $\hat{T}(\lambda, \xi)$ is uniformly bounded for all $\lambda \in \Sigma$ and $\xi \in \mathbb{R}^n$, with $|\lambda| + |\xi| \neq 0$.

Consider the function $\tilde{m}(\lambda, \xi) := m(\lambda, \xi)/\lambda$ given by

where $z_2 := \beta(\lambda)\xi^2/\lambda$. Let $\phi_j = \arg z_j$; then a short computation shows that

provided that $|\phi_1 - \phi_2| < \pi$. Here

From (A) and the Cauchy-Schwarz inequality we obtain

hence $|\phi| \leq \sigma < \pi/2$ for all $\xi \in \mathbb{R}^n$. Since $|\phi_2| = |\arg \lambda| \leq \phi$ we have

|\phi_1 - \phi_2| \leq \sigma + \phi < \pi,
provided $\phi > \pi/2$ is sufficiently close to $\pi/2$ and this in turn yields together with (A)
\[ |\tilde{m}(\lambda, \xi)| = |z_1 + z_2| \geq C(|z_1| + |z_2|) \geq C(1 + |\xi|^2 + |\xi|^4/|\lambda|) \]
or equivalently
\[ |m(\lambda, \xi)| \geq C(|\lambda|(1 + |\xi|^2) + |\xi|^4), \quad (4.14) \]
hence $|\tilde{T}(\lambda, \xi)| \leq C$ for all such $\lambda$ and $\xi$ from above.

In the next step we will verify (M) for $|\alpha| = 1$, uniformly in $\lambda \in \Sigma_\delta$. Observe that
\[ \partial_{\xi_j} \tilde{T}(\lambda, \xi) = \frac{m(\lambda, \xi)(2\lambda\xi_j + 4\xi_j|\xi|^2) - (\lambda(1 + |\xi|^2) + |\xi|^4)\partial_{\xi_j} m(\lambda, \xi)}{m(\lambda, \xi)^2}. \quad (4.15) \]
The derivative of $m(\lambda, \xi)$ is given by
\[ \partial_{\xi_j} m(\lambda, \xi) = \lambda \left( 2\beta(B\xi(e_j) - i(a_j + c_j) - a_j(c)\xi - c_j(a)\xi) + 2\beta \xi_j(B\xi(e_j) + (B\xi(e_j))|\xi|^2) \right), \]
and this yields
\[ |\partial_{\xi_j} m(\lambda, \xi)| \leq C(|\lambda|(1 + |\xi|) + |\xi|^3). \]
Young’s inequality implies furthermore that
\[ |m(\lambda, \xi)| \leq C(|\lambda|(1 + |\xi|^2) + |\xi|^4) \]
and thus we obtain from (4.14) and (4.15) the estimate
\[ |\partial_{\xi_j} \tilde{T}(\lambda, \xi)| \leq C \frac{|\lambda|(1 + |\xi|) + |\xi|^3}{|\lambda|(1 + |\xi|^2) + |\xi|^4}, \]
whence we see that
\[ |\xi||\partial_{\xi_j} \tilde{T}(\lambda, \xi)| \leq C < \infty, \]
for all $\lambda \in \Sigma_\phi$ and $\xi \in \mathbb{R}^n$, with $|\lambda| + |\xi| \neq 0$. Inductively it follows that (M) is fulfilled for each multiindex $\alpha \in \mathbb{N}_0^d$, uniformly in $\lambda \in \Sigma_\phi$. The classical Mikhlin multiplier theorem then implies that $T$ is a Fourier multiplier in $L_p(\mathbb{R}^n)$ w.r.t the variable $\xi$ and this yields a holomorphic uniformly bounded family $\{T(\lambda)\}_{\lambda \in \Sigma_\phi} \subseteq B(L_p(\mathbb{R}^n))$, $\phi > \pi/2$. By [15, Theorem 3.2] this family is also $\mathcal{R}$-bounded in $L_p(\mathcal{R} \times \mathbb{R}^n)$. Finally, since the operator $\partial_t$ admits a bounded $\mathcal{H}^\infty$-calculus with angle $\pi/2$ we obtain from Theorem 1.3.1 that $T$ is bounded in $L_p(J; L_p(\mathbb{R}^n))$. For the functions $u = (I - \Delta)^{1/2}w$ and $\mu = (I - \Delta)^{1/2}p$, this yields
\[ u \in H^1_p(J; H^1_p(\mathbb{R}^n)) \cap L_p(J; H^2_p(\mathbb{R}^n)), \]
and
\[ \mu \in L_p(J; H^2_p(\mathbb{R}^n)). \]
The proof is complete. \qed

It is possible to extend Theorem 4.2.1 to the case of variable coefficients with a small deviation from constant ones. To prove this result we write the coefficients in the form
\[ a(x) = a^0 + a^1(x), \quad c(x) = c^0 + c^1(x) \quad \text{and} \quad B(x) = B^0 + B^1(x), \]
where $|a^1|_{L_\omega(\mathbb{R}; \mathbb{R}^n)} + |c^1|_{L_\omega(\mathbb{R}; \mathbb{R}^n)} + |B^1|_{L_\omega(\mathbb{R}; \mathbb{R}^{n \times n})} \leq \omega$, with some constant $\omega > 0$ and $a^1, c^1 \in W^1_\omega(\mathbb{R}^n; \mathbb{R}^n), B^1 \in W^1_\omega(\mathbb{R}^n; \mathbb{R}^{n \times n})$. Furthermore we require $\text{div } a^1(x) = \text{div } c^1(x) = 0$ for all $x \in \mathbb{R}^n$ and we assume that $(\beta, a^0, c^0, B^0)$ satisfy condition (A). Then also $(\beta, a(x), c(x), B(x))$ satisfy (A) with a possibly smaller constant $\varepsilon > 0$, provided $\omega > 0$ is sufficiently small. Note that due to the uniform boundedness of the data, the norms of the solution operators are uniform as well. Therefore we may cut the interval $J = [0, T]$ into pieces $J_i = [i\delta, i\delta + \delta]$ for some $\delta > 0$. The proof is complete. \qed
We then solve the problem successively on $J_t$. W.l.o.g. we may treat the first interval $J_0$. For this purpose we define the spaces $Z^j_\delta$, $X^j_\delta$, $j = 1, 2$ as the restriction of the spaces $Z^j$, $X^j$ to the interval $J_0$. Note that w.l.o.g. we may assume $u_0 = 0$

Let $S$ denote the solution operator for the constant coefficient case from Theorem 4.2.1 and denote by $T$ that of the perturbed problem. Assume that we already know a solution to the perturbed problem. Then it is easy to verify the identity

$$T = S + SBT,$$

where $B \left( \begin{array}{c} u \\ \mu \end{array} \right) = \left[ \begin{array}{c} \text{div}(a^1(x)\partial_xu) + \text{div}(B^1(x)\nabla\mu) \\ c^1(x) \cdot \nabla\mu \end{array} \right],$$

(4.16)

and $(u, \mu) \in 0Z^1_\delta \times Z^2_\delta$ is the solution of the perturbed problem. From the assumption on the coefficients we obtain the following estimate

$$\left| B \left( \begin{array}{c} u \\ \mu \end{array} \right) \right|_{X^3_\delta \times X^2_\delta} \leq C \left( |u|^\infty |u|_{Z^3_\delta} + (|B^1|_\infty + |c^1|_\infty)|\mu|_{Z^2_\delta} + |\partial_t u|_{L_p(J_0 \times \Omega)} + |\mu|_{L_p(J_0; H^1_0(\Omega))} \right).$$

(4.17)

The task is to estimate the terms $|\partial_t u|_{L_p(J_0 \times \Omega)}$ and $|\mu|_{L_p(J_0; H^1_0(\Omega))}$, since they are not of lower order with respect to the variable $t$. To this end we consider the elliptic problem

$$\mu - (a + c) \cdot \nabla \mu + \text{div}(a(c \cdot \nabla \mu)) - \text{div}(\beta B \nabla \mu) = \text{div}(a\Delta u) - \Delta u + \tilde{f},$$

which results, if we replace $\partial_t u$ in (4.11)_1 by the second equation in (4.11)_2, where

$$\tilde{f} := \beta f + a \cdot \nabla g - g \in L_p(J \times \mathbb{R}^n)$$

is a fixed function. For this elliptic problem we obtain the following a priori estimate.

**Proposition 4.2.2.** There exists a constant $M > 0$ such that

$$|\mu|_{L_p(J_0; H^1_0(\mathbb{R}^n))} + |\partial_t u|_{L_p(J_0; L_p(\mathbb{R}^n))} \leq M(|u|_{L_p(J_0; H^2_0(\mathbb{R}^n))} + |\tilde{f}|_{L_p(J_0; L_p(\mathbb{R}^n))} + |g|_{L_p(J_0; L_p(\mathbb{R}^n))}).$$

**Proof.** First we show that the $L_p$-realization $A_0$ of the differential operator

$$A_0(D) = c \cdot \nabla w + a \cdot \nabla w - \text{div}(a(c \cdot \nabla w)) + \text{div}(\beta B \nabla w)$$

with domain $D(A_0) = L_p(J_0; H^2_0(\mathbb{R}^n))$ is dissipative. To this end, we compute

$$\text{Re} \int_{\mathbb{R}^n} A_0 w \bar{w} |w|^{p-2} \, dx$$

$$= \text{Re} \left( \int_{\mathbb{R}^n} (c + a) \cdot \nabla w \bar{w} |w|^{p-2} \, dx + \int_{\mathbb{R}^n} (\text{div}(\beta B \nabla w) - \text{div}(a(c \cdot \nabla w)))|w|^{p-2} \, dx \right)$$

$$= \frac{1}{p} \int_{\mathbb{R}^n} (c + a) \nabla |w|^p \, dx - \text{Re} \int_{\mathbb{R}^n} (\beta B \nabla w - a(c \cdot \nabla w)) \cdot \nabla |w|^{p-2} \, dx$$

$$= - \int_{\mathbb{R}^n} |w|^{p-4} \text{Re} \left( \frac{p}{2} (\tilde{B} \nabla w \cdot \nabla \bar{w}) |w|^2 + \left( \frac{p}{2} - 1 \right) (\tilde{B} \nabla w \cdot \nabla w) \bar{w}^2 \right) \, dx$$

for each $w \in H^2_0(\mathbb{R}^n)$, where $\tilde{B} : = \beta B - \frac{1}{4}(a \otimes c + c \otimes a)$. Here we used integration by parts, and the fact that $\text{div} a(x) = \text{div} c(x) = 0$. To estimate the integral, we set $\nabla w = u + iv$ and $w = b_1 + ib_2$, with $u, v \in \mathbb{R}^n$ and $b_1, b_2 \in \mathbb{R}$. This yields

$$\text{Re} \left( \frac{p}{2} (\tilde{B} \nabla w \cdot \nabla \bar{w}) |w|^2 + \left( \frac{p}{2} - 1 \right) (\tilde{B} \nabla w \cdot \nabla w) \bar{w}^2 \right)$$

$$= \frac{p}{2} (\tilde{B} u |u|^2 + |b_2|^2) + \frac{p}{2} (\tilde{B} v |v|^2 + |b_1|^2) + \left( \frac{p}{2} - 1 \right) (\tilde{B} u |b_1|^2 + |b_2|^2)$$

$$- \left( \frac{p}{2} - 1 \right) (\tilde{B} v |b_1|^2 + |b_2|^2) + 4 \left( \frac{p}{2} - 1 \right) (\tilde{B} u |b_1| b_2)$$

$$= (p - 1)(\tilde{B} u |u|^2 + |b_2|^2) + (\tilde{B} u |b_2|^2 + |b_1|^2) + (p - 1)(\tilde{B} v |v|^2 + |b_1|^2) + 2(p - 2)(\tilde{B} u |b_1| b_2)$$

$$= (p - 1) \left( (\tilde{B} u |u|^2 + (\tilde{B} v |v|^2 + 2(\tilde{B} u |b_1| b_2) + (\tilde{B} u |b_2|^2 + (\tilde{B} v |v|^2 + (2(\tilde{B} u |b_1| b_2) + |b_2|^2 b_1 b_2.}$$
Thanks to the Cauchy-Schwarz inequality, Young’s inequality and Assumption (A), it follows that $\langle Bu|u\rangle b_1^2 + \langle Bv|v\rangle b_2^2 + 2\langle Bu|v\rangle b_1 b_2$ is nonnegative. To see this, we estimate as follows

$$|2\langle Bu|v\rangle b_1 b_2| \leq 2\sqrt{\langle Bu|u\rangle} \sqrt{\langle Bv|v\rangle} |b_1||b_2| \leq b_1^2 \langle Bu|u\rangle + b_2^2 \langle Bv|v\rangle.$$  

By the same arguments, the term $\langle Bu|u\rangle b_1^2 + \langle Bv|v\rangle b_2^2 - 2\langle Bu|v\rangle b_1 b_2$ is nonnegative too. This yields

$$\text{Re} \left\{ \frac{p}{2} \langle B\nabla w \cdot \nabla \bar{w} \rangle |w|^2 + \left( \frac{p}{2} - 1 \right) \langle B\nabla w \cdot \nabla w \rangle \bar{w}^2 \right\}$$

$$\geq \min \{1, (p - 1)\} \left\{ (\langle Bu|u\rangle + \langle Bv|v\rangle)(b_1^2 + b_2^2) \right\}$$

$$\geq \epsilon \min \{1, (p - 1)\} \{|u|^2 + |v|^2\}(b_1^2 + b_2^2) = \epsilon \min \{1,(p - 1)\} |\nabla w|^2 |w|^2,$$

by condition (A). This shows that $A_0$ is dissipative. Next we split the operator $A_0(D) = A_0^\#(D) + A_0^{\mathrm{ow}}(D)$, with

$$A_0^\#(D)w = \beta B : \nabla^2 w - (\nabla^2 w)c \cdot a,$$

and

$$A_0^{\mathrm{ow}}(D)w = \beta \text{div} B \cdot \nabla w - \nabla \text{div} w \cdot a + (a + c) \cdot \nabla w,$$

where we used again the property $\text{div} a(x) = 0$. Here $\text{div} A$ denotes the divergence of a matrix $A$, defined by

$$\text{Div} A = \left( \sum_{j=1}^n \partial_j (a_{ij}) \right)_{i=1, \ldots, n} \in \mathbb{R}^n.$$  

Furthermore we use the notation $B : \nabla^2 w = \sum_{i,j} b_{ij} \partial_i \partial_j w$. By condition (A) it is easily seen that the principal part $A_0^\#(D)$ of $A_0(D)$ is parameter elliptic in the sense of [12, Definition 5.1]. Note that the coefficients in the lower order terms are smooth. By [12, Theorem 5.7] there exists some $\lambda > 0$ such that $\lambda - A_0$ is $\mathcal{R}$-sectorial, hence also sectorial. This in turn yields that $A_0$ is the generator of a contraction semigroup in $L_p(J_0; L_p(\mathbb{R}^n))$, by the Lumer-Phillips Theorem and the dissipativity of $A_0$. In particular the operator $(I - A_0)$ is invertible. Consider the equation

$$\mu_1 - A_0(D)\mu_1 = -\Delta u + f.$$  

(4.18)

By the above considerations the solution $\mu_1 \in L_p(J_0; H^{2}_{\mu}(\mathbb{R}^n))$ of (4.18) is unique and satisfies the estimate $|\mu_1|_{L_p(J_0; H^{2}_{\mu}(\mathbb{R}^n))} \leq C|u|_{L_p(J_0; H^{2}_{\mu}(\mathbb{R}^n))} + |f|_{L_p(J_0; L_p(\mathbb{R}^n))}$ for some constant $C > 0$. Then the function $\mu_2 = \mu - \mu_1$ solves the equation

$$\mu_2 - A_0(D)\mu_2 = \text{div}(a\Delta u).$$  

(4.19)

In a next step, we want to write $\mu_2 = \text{div} \mu_3 + \mu_4$ for some suitable functions $\mu_3, \mu_4$. To this end, we consider firstly the following equations

$$\mu_3^j - A_0(D)\mu_3^j = a_j \Delta u, \quad j \in \{1, \ldots, n\},$$  

(4.20)

where $a_j$ is the $j$th component of the vector $a$. Each equation admits a unique solution $\mu_3^j \in L_p(J_0; H^{2}_{\mu}(\mathbb{R}^n))$ and we have the estimate

$$|\mu_3^j|_{L_p(J_0; H^{2}_{\mu}(\mathbb{R}^n))} \leq C|u|_{L_p(J_0; H^{2}_{\mu}(\mathbb{R}^n))}$$

for each $j \in \{1, \ldots, n\}$ and some constant $C > 0$ at our disposal. Setting $\mu_3 = [\mu_3^1, \ldots, \mu_3^n]^T$ and applying the divergence operator to the system of equations (4.20), we obtain

$$\text{div} \mu_3 - A_0(D)(\text{div} \mu_3) = \text{div} (a\Delta u) - [A_0(D), \text{div}]\mu_3,$$
where \([A_0(D), \text{div}]\mu_3\) denotes the commutator of \(A_0(D)\) and \(\text{div}\), i.e.
\[ [A_0(D), \text{div}]\mu_3 := A_0(D)(\text{div} \mu_3) - \text{div}(A_0(D)\mu_3), \]
which is in fact an operator of second order. Let \(\mu_4\) denote the unique solution of
\[ \mu_4 - A_0(D)\mu_4 = -[A_0(D), \text{div}]\mu_3, \]
with the estimate
\[ |\mu_4|_{L_p(J_0;H^2_\delta(\mathbb{R}^n))} \leq C|[A_0(D), \text{div}]\mu_3|_{L_p(J_0;L_p(\mathbb{R}^n))} \leq C|\mu|_{L_p(J;H^2_\delta(\mathbb{R}^n))}, \]
by the estimate for \(\mu_3\). Finally, by the uniqueness of the solution \(\mu_2\) of (4.19) we may conclude that \(\mu_2 = \text{div} \mu_3 + \mu_4\). This in turn yields the desired estimate for \(\mu\), since \(\mu = \mu_1 + \text{div} \mu_3 + \mu_4\).

To estimate \(\partial_t u\) in \(L_p(J; L_p(\mathbb{R}^n))\), we make use of equation (4.11). This completes the proof.

Now we go back to (4.17) to obtain the estimate
\[ \left| B \begin{bmatrix} u \\ \mu \end{bmatrix} \right|_{X_1^1 \times X_2^1} \leq C \left( |a|_\infty |u|_{Z_1^3} + (|B^2|_\infty + |c^1|_\infty) |\mu|_{Z_3^4} \right. \]
\[ + \left| u \right|_{L_p(J_0;H^2_\delta(\mathbb{R}^n))} + \left| f \right|_{L_p(J_0;L_p(\mathbb{R}^n))} + |g|_{L_p(J_0;H^2_\delta(\mathbb{R}^n))} \right). \]

We use the mixed derivative theorem to obtain
\[ Z_1^3 = H^1_p(J_0; H^1_p(\mathbb{R}^n)) \cap L_p(J_0; H^3_p(\mathbb{R}^n)) \subseteq H^{1/2}_p(J_0; H^2_p(\mathbb{R}^n)) \subseteq L_2(J_0; H^2_p(\mathbb{R}^n)). \]
This in turn yields
\[ |u|_{L_p(J_0;H^2_\delta(\mathbb{R}^n))} \leq C^{1/2p} |u|_{L_2(J_0;H^2_\delta(\mathbb{R}^n))} \leq C^{1/2p} |u|_{Z_1^3}. \]

If we choose \(\delta > 0\) and \(\omega > 0\) small enough, we obtain from (4.16) the estimate
\[ |(u, \mu)|_{Z_1^1 \times Z_2^1} \leq M(|f|_{X_1^1} + |g|_{X_2^1} + |u_0|_{X_3^I}), \]
for the solution of the perturbed problem. Therefore the operator \(L \in B(Z_1^3 \times Z_4^1; X_1^2 \times X_2^2 \times X_p)\) which is defined by the first lines of the left hand side of (4.11) is injective and has closed range, i.e. it is a semi Fredholm operator. To show surjectivity of \(L\) we apply a continuation argument for semi Fredholm operators, which is due to KATÔ [21]. Let \(L_\tau\) be the corresponding operator to (4.11) with data
\[ (\beta_\tau, a_\tau, c_\tau, B_\tau) := (1 - \tau)(\beta, a^0, c^0, B^0) + \tau(\beta, a, c, B), \quad \tau \in [0, 1]. \]

By Theorem 4.2.1 the operator \(L_0\) is bijective, since the data \((\beta, a^0, c^0, B^0)\) satisfy Assumption (A). Furthermore, the data \((\beta_\tau, a_\tau, c_\tau, B_\tau)\) satisfy (A) too, by the smallness of \(\omega > 0\). It is also clear that \(a_\tau\) and \(c_\tau\) are divergence free vector fields and \((a_\tau, c_\tau, B_\tau)\) enjoy the same regularity as \((a, c, B)\). Hence each operator \(L_\tau\) is injective and has closed range, by the above calculation. Finally, the continuity property of the Fredholm index yields that the index of \(L_\tau\) is zero for each \(\tau \in [0, 1]\). This proves that \(L_1 = L\) is also surjective. Therefore we have the following result.

**Corollary 4.2.3.** Let \(a_1, c_1 \in W^{1}_\infty(\mathbb{R}^n; \mathbb{R}^n)\) and \(B_1 \in W^{1}_\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})\) with \(\text{div} a_1(x) = \text{div} c_1(x) = 0\) for all \(x \in \mathbb{R}^n\). Then Theorem 4.2.1 remains valid in case of variable coefficients
\[ a(x) = a^0 + a^1(x), \quad c(x) = c^0 + c^1(x) \quad \text{and} \quad B(x) = B^0 + B^1(x), \]
provided that \((\beta, a^0, c^0, B^0)\) satisfy (A) and
\[ |a^1|_{L_\infty(\mathbb{R}^n; \mathbb{R}^n)} + |c^1|_{L_\infty(\mathbb{R}^n; \mathbb{R}^n)} + |B^1|_{L_\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \leq \omega, \]
with \(\omega > 0\) being sufficiently small.
4.3 The Linear Cahn-Hilliard-Gurtin Problem in $\mathbb{R}^n$

Set $x = (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ and consider the half space problem

$$
\begin{aligned}
\partial_t u - \text{div}(a \partial_t u) &= \text{div}(B \nabla \mu) + f, \quad t \in J, \ x' \in \mathbb{R}^{n-1}, \ y > 0, \\
\mu - c \cdot \nabla \mu &= \beta \partial_t u - \Delta u + g, \quad t \in J, \ x' \in \mathbb{R}^{n-1}, \ y > 0, \\
B \nabla \mu \cdot \nu &= h_1, \quad t \in J, \ x' \in \mathbb{R}^{n-1}, \ y = 0, \\
\partial_y u &= h_2, \quad t \in J, \ x' \in \mathbb{R}^{n-1}, \ y = 0,
\end{aligned}
$$

(4.21)

where $J = [0, T]$, $\nu$ is the outer unit normal at $x \in \partial \mathbb{R}_+^n$, i.e. $\nu = [0, \ldots, 0, -1]^T$, and the data $(\beta, a, c, B)$ are subject to Assumption (A). Due to the conditions (4.9) and (4.10) it holds that $a = (a_0, 0) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $c = (c_0, 0) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and

$$
B = \begin{bmatrix} B_0 & 0 \\ 0 & B_2 \end{bmatrix},
$$

where $B_0 \in \mathbb{R}^{(n-1) \times (n-1)}$ is symmetric and $B_2 \in \mathbb{R}$. We assume that $B$ is positive definite, hence $B_2 > 0$.

The main result on optimal regularity of (4.21) reads as follows.

**Theorem 4.3.1.** Let $1 < p < \infty$ and assume that (A) and (4.9) and (4.10) hold true. Then (4.21) admits a unique solution

$$
\begin{aligned}
u &\in L_p(J; H^1_p(\mathbb{R}_+^n)) \cap L_p(J; H^2_p(\mathbb{R}_+^n)) =: Z^1, \\
\mu &\in L_p(J; H^2_p(\mathbb{R}_+^n)) =: Z^2,
\end{aligned}
$$

if and only if the data is subject to the following conditions.

(i) $f \in L_p(J; L_p(\mathbb{R}_+^n)) =: X^1$,

(ii) $g \in L_p(J; H^1_p(\mathbb{R}_+^n)) =: X^2$,

(iii) $h_1 \in L_p(J; W^{1-1/p}_p(\mathbb{R}^{n-1})) =: Y^1$,

(iv) $h_2 \in W^{1-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^{n-1})) =: Y^2$,

(v) $u_0 \in B^{1-2/p}_p(\mathbb{R}^n) =: X_p$,

(vi) $\partial_y u_0 = h_2|_{y=0}$ if $p > 3/2$.

**Proof.** The necessity part follows from the equations and trace theory, cf. Theorem 1.4.3, so we can turn to the sufficiency part. We want to remark that due to the structure of the matrix $B$, the boundary condition $B \nabla \mu \cdot \nu|_{y=0}$ becomes

$$
B \nabla \mu \cdot \nu|_{y=0} = B_2 \partial_y \mu|_{y=0},
$$

with $B_2 > 0$ since $B$ is assumed to be positive definite. Now we want to set $h_1 = h_2 = u_0 = 0$. For this purpose we first solve the elliptic problem

$$
\begin{aligned}
(I - \Delta_x) \eta - \partial_y^2 \eta &= 0, \quad x' \in \mathbb{R}^{n-1}, \ y > 0, \\
\partial_y \eta &= h_1/B_2, \quad x' \in \mathbb{R}^{n-1}, \ y = 0.
\end{aligned}
$$

(4.22)

Define $\tilde{L} := (I - \Delta_x)^{1/2}$ in $L_p(\mathbb{R}^{n-1})$, with $D(\tilde{L}) = H^1_0(\mathbb{R}^{n-1})$ and let $L$ denote the natural extension of $\tilde{L}$ to $L_p(J; L_p(\mathbb{R}^{n-1}))$, that is $D(L) = L_p(J; H^1_p(\mathbb{R}^{n-1}))$ and $Lu = \tilde{L}u$ for each $u \in D(L)$. Then the unique solution $\eta$ of (4.22) is given by

$$
\eta(y) = -L^{-1} e^{-L_y}(h_1/B_2).
$$
To this end we extend move
On the other hand, if we consider the function \( v \) where \( \tilde{v} \) denotes the restriction of \( \tilde{\eta} \) and therefore \( \eta \in L_p(J; H^1_p(\mathbb{R}_+^n)) \), with \( B_2 \partial_y \eta|_{y=0} = h_1 \). In order to remove \( h_2 \) and \( u_0 \), we solve the initial boundary value problem
\[
\beta \partial_t v - \Delta_{x'} v - \partial_y^2 v = 0, \quad t \in J, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0,
\]
\[
\partial_y v = h_2, \quad t \in J, \quad x' \in \mathbb{R}^{n-1}, \quad y = 0,
\]
\[
v(0) = u_0, \quad t = 0, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0.
\]

To this end we extend \( u_0 \in B^{3-2/p}_{pp}(\mathbb{R}_+^n) \) to a function \( \tilde{u}_0 \in B^{3-2/p}_{pp}(\mathbb{R}^n) \) and solve the heat equation
\[
\beta \partial_t \tilde{u} - \Delta \tilde{u} = 0, \quad t \in J, \quad x \in \mathbb{R}^n, \quad \tilde{u}(0) = \tilde{u}_0, \quad t = 0, \quad x \in \mathbb{R}^n,
\]
in \( L_p(J; H^1_p(\mathbb{R}^n)) \). This yields a solution
\[
\tilde{v} \in H^1_p(J; H^1_p(\mathbb{R}^n)) \cap L_p(J; H^3_p(\mathbb{R}^n)).
\]
If \( v_1 := P\tilde{v} \) denotes the restriction of \( \tilde{v} \) to the half space \( \mathbb{R}_+^n \), the function \( v_2 := v - v_1 \) should solve the initial boundary value problem
\[
\beta \partial_t v_2 - \Delta_{x'} v_2 - \partial_y^2 v_2 = 0, \quad t \in J, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0,
\]
\[
\partial_y v_2 = \tilde{h}_2, \quad t \in J, \quad x' \in \mathbb{R}^{n-1}, \quad y = 0,
\]
\[
v_2(0) = 0, \quad t = 0, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0,
\]

where \( \tilde{h}_2 := h_2 - \partial_y v_1|_{y=0} \). Set \( v_3 = (I - \Delta_{x'})^{1/2} v_2 \). Then \( v_3 \) is a solution of
\[
\beta \partial_t v_3 - \Delta_{x'} v_3 - \partial_y^2 v_3 = 0, \quad t \in J, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0,
\]
\[
\partial_y v_3 = h_3, \quad t \in J, \quad x' \in \mathbb{R}^{n-1}, \quad y = 0,
\]
\[
v_3(0) = 0, \quad t = 0, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0,
\]

with \( h_3 = (I - \Delta_{x'})^{1/2} \tilde{h}_2 \in 0W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W^{1/2-1/p}_p(\mathbb{R}^{n-1})) \). We define \( L = (\beta \partial_t - \Delta_{x'})^{1/2} \) with natural domain
\[
D(L) = 0H^{1/2}_p(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; H^2_p(\mathbb{R}_+^{n-1})).
\]
Then, the unique solution \( v_3 \) of (4.25) is given by
\[
v_3(y) = -L^{-1}e^{-Ly}h_3,
\]
and \( h_3 \in D_L(1 - 1/p, p) \). This yields
\[
v_3 \in 0H^1_p(J; L_p(\mathbb{R}_+)) \cap L_p(J; H^2_p(\mathbb{R}_+^n)).
\]
On the other hand, if we consider the function \( v_4 := \partial_y v_2 \) as the solution of
\[
\beta \partial_t v_4 - \Delta_{x'} v_4 - \partial_y^2 v_4 = 0, \quad t \in J, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0,
\]
\[
v_4(0) = 0, \quad t = 0, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0,
\]
we obtain \( v_4(y) = e^{-Ly}h_2 \) and \( h_2 \in D_L(2 - 1/p, p) \). This yields
\[
v_4 \in 0H^1_p(J; L_p(\mathbb{R}_+)) \cap L_p(J; H^2_p(\mathbb{R}_+^n)).
\]
From the regularity of \( v_3 \) and \( v_4 \) we may conclude that
\[
v_2 \in 0H^1_p(J; H^1_p(\mathbb{R}_+^n)) \cap L_p(J; H^3_p(\mathbb{R}_+^n)).
\]
4.3. The Linear Cahn-Hilliard-Gurtin Problem in $\mathbb{R}^n$

Now the functions $u_1 := u - v$ and $\mu_1 := \mu - \eta$, with $v = v_1 + v_2$, should solve the system

$$\begin{align*}
\partial_t u_1 - \text{div}(a\partial_t u_1) &= \text{div}(B\nabla \mu_1) + f_1, & t \in J, \ x' \in \mathbb{R}^{n-1}, \ y > 0, \\
\mu_1 - c \cdot \nabla \mu_1 &= \beta \partial_t u_1 - \Delta u_1 + g_1, & t \in J, \ x' \in \mathbb{R}^{n-1}, \ y > 0, \\
B_2 \partial_y \mu_1 &= 0, & t \in J, \ x' \in \mathbb{R}^{n-1}, \ y = 0, \\
\partial_y u_1 &= 0, & t \in J, \ x' \in \mathbb{R}^{n-1}, \ y = 0, \\
u_1(0) &= 0, & t = 0, \ x' \in \mathbb{R}^{n-1}, \ y > 0,
\end{align*}$$

(4.27)

with some modified data $f_1 \in X^1$ and $g_1 \in X^2$. In a next step we extend the functions $f_1$ and $g_1$ to $J \times \mathbb{R}^n$ by even reflection, i.e. we set

$$
\begin{align*}
f_2(t, x', y) &= \begin{cases} f_1(t, x', y), & \text{if } y \geq 0 \\
f_1(t, x', -y), & \text{if } y \leq 0 \end{cases} & \text{and} &
\begin{align*}
g_2(t, x', y) &= \begin{cases} g_1(t, x', y), & \text{if } y \geq 0 \\
g_1(t, x', -y), & \text{if } y \leq 0. \end{cases}
\end{align*}
\end{align*}
$$

Thanks to Theorem 4.2.1 we can solve the full space problem

$$\begin{align*}
\partial_t u_2 - \text{div}(a\partial_t u_2) &= \text{div}(B\nabla \mu_2) + f_2, & t \in J, \ x \in \mathbb{R}^n, \\
\mu_2 - c \cdot \nabla \mu_2 &= \beta \partial_t u_2 - \Delta u_2 + g_2, & t \in J, \ x \in \mathbb{R}^n, \\
u_2(0) &= 0, & t = 0, \ x \in \mathbb{R}^n,
\end{align*}$$

(4.28)

since $f_2 \in L_p(J \times \mathbb{R}^n)$ and $g_2 \in L_p(J; H^1_l(\mathbb{R}^n))$. This yields a unique solution

$$u_2 \in H^1_p(J; H^1_l(\mathbb{R}^n)) \cap L_p(J; H^2_p(\mathbb{R}^n)) \quad \text{and} \quad \mu_2 \in L_p(J; H^2_p(\mathbb{R}^n)),$$

by Theorem 4.2.1. At this point we emphasize that the equations (4.27)$_{1,2}$ are invariant w.r.t. even reflection in the variable $y$, since $a_1 = c_1 = 0$ and $B_1 = 0$. This in turn implies that the solution $(u_2, \mu_2)$ is symmetric, w.r.t. the variable $y$ and this yields necessarily, $\partial_y u_2 |_{y=0} = \partial_y \mu_2 |_{y=0} = 0$. Denoting by $P$ the restriction of the solution $(u_2, \mu_2)$ to the half space $\mathbb{R}^n_+$, it follows that $(u_1, \mu_1) = P(u_2, \mu_2)$ is the unique solution of (4.27) and therefore $u = u_1$ and $\mu = \mu_1$ is the unique solution of (4.21). The proof is complete. $\square$

As in Section 4.2, we may extend Theorem 4.3.1 to the case of variable coefficients with a small deviation from constant ones. The arguments are similar to those in the proof of Corollary 4.2.3. Indeed it suffices to show that there is a version of Proposition 4.2.2 for the half space case. Assume that we have given coefficients

$$a(x) = a^0 + a^1(x), \quad c(x) = c^0 + c^1(x) \quad \text{and} \quad B(x) = B^0 + B^1(x),$$

where $|a^1|_{L^\infty(\mathbb{R}^n_+; \mathbb{R}^n)} + |c^1|_{L^\infty(\mathbb{R}^n_+; \mathbb{R}^n)} + |B^1|_{L^\infty(\mathbb{R}^n_+; \mathbb{R}^{n \times n})} \leq \omega$, with some constant $\omega > 0$ and $a^1, c^1 \in W^1_\infty(\mathbb{R}^n_+; \mathbb{R}^n)$, $B^1 \in W^1_\infty(\mathbb{R}^n_+; \mathbb{R}^{n \times n})$. Furthermore we require $\text{div} a^1(x) = \text{div} c^1(x) = 0$ for all $x \in \mathbb{R}^n_+$, 

$$(a^0|e_n) = (a^1(x)|e_n) = (c^0|e_n) = (c^1(x)|e_n) = 0$$

for all $x \in \partial \mathbb{R}^n_+$, and we assume that $(\beta, a^0, c^0, B^0)$ satisfy condition (A). Here $e_n = [0, \ldots, 0, -1]^T$. Finally, let $B^0$ satisfy (4.10). Extending the data to the whole of $\mathbb{R}^n$ we may w.l.o.g. assume that $f = g = u_0 = 0$. Then we have the following result.

**Proposition 4.3.2.** There exists a constant $M > 0$ such that

$$|u|_{L_p(J_0; H^1_l(\mathbb{R}^n_+))} + |\partial_t u|_{L_p(J_0; L_p(\mathbb{R}^n_+))} \leq M(|u|_{L_p(J_0; H^2_l(\mathbb{R}^n_+))} + |h_1|_{L_p(J_0; W^{1-1/p}_p(\mathbb{R}^{n-1}))}).$$

*Proof.* Extending the data $f, g$ and $u_0$ to the whole of $\mathbb{R}^n$ and solving the full space problem with Corollary 4.2.3 we may assume that $f = g = u_0 = 0$. The corresponding elliptic boundary value problem for $\mu$ reads

$$\mu - c \cdot \nabla \mu - a \cdot \nabla \mu + \text{div}(a(c \cdot \nabla \mu)) - \text{div}(\beta B \nabla \mu) = \text{div}(a \Delta u) - \Delta u, \quad B \nabla \mu \cdot \nabla \nu = h_1.$$
It is not difficult to show that the $L_p$-realization $A_0$ of the differential operator

$$A_0(D)w = c \cdot \nabla w + a \cdot \nabla w - \text{div}(a(c \cdot \nabla w)) + \text{div}(\beta B \nabla w)$$

with domain

$$D(A_0) = \{ v \in L_p(J_0; H^{2}_0(\mathbb{R}^n_+)) : B \nabla v \cdot \nu = 0 \},$$

is dissipative in $L_p(J_0; L_p(\mathbb{R}^n_+))$. In fact, we may exactly follow the lines of the proof of Proposition 4.2.2 since there appear no boundary terms. This is due to the boundary conditions $B \nabla v \cdot \nu = 0$, $v \in D(A_0)$, and (4.9). To prove invertibility of $I - A_0$ we use following identity.

$$\text{div}(a(c \cdot \nabla \mu)) = a \cdot \nabla (c \cdot \nabla \mu) = a \cdot (\nabla c \nabla \mu) + a \cdot (\nabla^2 \mu)$$

$$= a \cdot (\nabla c \nabla \mu) + \frac{1}{2} (a \otimes c + c \otimes a) \nabla^2 \mu$$

$$= a \cdot (\nabla c \nabla \mu) - \frac{1}{2} [\text{div}(a \otimes c + c \otimes a)] \cdot \nabla \mu + \frac{1}{2} \text{div}(a \otimes c + c \otimes a) \nabla \mu].$$

Owing to this identity, we may write $A_0(D) = A_1(D) + A_1^{\text{low}}(D)$, with

$$A_1(D) \mu = \text{div}(\tilde{B} \nabla \mu),$$

and

$$A_1^{\text{low}}(D) \mu = \frac{1}{2} [\text{div}(a \otimes c + c \otimes a)] \cdot \nabla \mu - a \cdot (\nabla c \nabla \mu) + (a + c) \cdot \nabla \mu.$$

Here the matrix $\tilde{B}$ is given by $\beta B - \frac{1}{2} (a \otimes c + c \otimes a)$. Observe that

$$\tilde{B} \nabla \mu \cdot \nu = \nabla \mu \cdot B \nu = B \nabla \mu \cdot \nu = \gamma_1,$$

by the assumption (4.9) on the vector fields $a, c$ and since the matrices $B$ and $\tilde{B}$ are symmetric. Consider the linear elliptic problem with a conormal boundary condition

$$w - \text{div}(\tilde{B} \nabla w) = f, \ (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R}_+, \ w \in L^1(\mathbb{R}^{n-1} \times \mathbb{R}_+),$$

$$\tilde{B} \nabla w \cdot \nu = g, \ x' \in \mathbb{R}^{n-1}, \ y = 0.$$  \hspace{1cm} (4.30)

Elliptic problems of this type have been extensively studied in the literature and it is well-known that (4.30) admits a unique solution $w \in H^{2}_p(\mathbb{R}^n_+)$ if and only if $(f, g) \in L_p(\mathbb{R}^n_+) \times W^{1-1/p}_p(\mathbb{R}^{n-1})$. In addition, there exists a constant $M > 0$ such that the estimate

$$|w|_{L_p(J_0; H^{2}_0(\mathbb{R}^n_+))} \leq M (|f|_{L_p(J_0; L_p(\mathbb{R}^n_+))} + |g|_{L_p(J_0; W^{1-1/p}_p(\mathbb{R}^{n-1}))})$$

holds, i.e. we have maximal regularity of type $L_p$ for (4.30). Then, by perturbation theory, there exists $\lambda_0 \geq 0$ such that

$$(1 + \lambda_0) w - A_1^{\text{low}}(D) w - \text{div}(\tilde{B} \nabla w) = f, \ (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R}_+, \ w \in L^1(\mathbb{R}^{n-1} \times \mathbb{R}_+),$$

$$\tilde{B} \nabla w \cdot \nu = g, \ x' \in \mathbb{R}^{n-1}, \ y = 0, \hspace{1cm} (4.31)$$

has a unique solution $w \in H^{2}_p(\mathbb{R}^n_+)$ if and only if $(f, g) \in L_p(\mathbb{R}^{n-1}) \times W^{1-1/p}_p(\mathbb{R}^{n-1})$. Setting $g = 0$, this yields the invertibility of the operator $I - A_0$, since $A_0$ is dissipative. Here we used the canonical extension of the differential operators from the basic space $L_p(\mathbb{R}^n_+)$ to $L_p(J_0; L_p(\mathbb{R}^n_+))$.

After these considerations we go back to (4.29). For $h_1 \in L_p(J_0; W^{1-1/p}_p(\mathbb{R}^{n-1}))$, let $\mu_1 \in L_p(J_0; H^{2}_p(\mathbb{R}^n_+))$ be the unique solution of the boundary value problem

$$\mu_1 - \text{div}(B \nabla \mu_1) = 0, \ x' \in \mathbb{R}^{n-1}, \ y > 0,$$

$$B \nabla \mu_1 \cdot \nu = h_1, \ x' \in \mathbb{R}^{n-1}, \ y = 0,$$

with the estimate

$$|\mu_1|_{L_p(J_0; H^{2}_p(\mathbb{R}^n_+))} \leq C|h_1|_{L_p(J_0; W^{1-1/p}_p(\mathbb{R}^{n-1}))},$$
4.3. The Linear Cahn-Hilliard-Gurtin Problem in $\mathbb{R}^n_+$

for some constant $C > 0$. Then the function $\mu_2 := \mu - \mu_1$ is the unique solution of
\[ \mu_2 - \mathcal{A}_0(D)\mu_2 = \text{div}(a \Delta u) - \Delta u - (I - \mathcal{A}_0(D))\mu_1, \quad B \nabla \mu_2 \cdot \nu = 0. \]
Define the function $\mu_3 \in L_p(J; H^2_p(\mathbb{R}^n_+))$ to be the unique solution of
\[ \mu_3 - \mathcal{A}_0(D)\mu_3 = -\Delta u - (I - \mathcal{A}_0(D))\mu_1, \quad B \nabla \mu_3 \cdot \nu = 0, \]
subject to the estimate
\[ |\mu_3|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))} \leq C|u|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))} + |h_1|_{L_p(J_0; W^{1,p}_{-1/p}(\mathbb{R}^{n-1}))}, \]
for some constant $C > 0$. Then the new function $\mu_4 := \mu_2 - \mu_3$ solves
\[ \mu_4 - \mathcal{A}_0(D)\mu_4 = \text{div}(a \Delta u), \quad B \nabla \mu_4 \cdot \nu = 0. \]

Define the boundary operator $\mathcal{B}(D)$ by
\[ \mathcal{B}(D)v = B \nabla v \cdot \nu. \]

Then we solve the system of equations
\[ \mu_4^j - \mathcal{A}_0(D)\mu_4^j = a_j \Delta u, \quad \mathcal{B}(D)\mu_4^j = 0, \quad (4.32) \]
for each $j = 1, \ldots, n$ to obtain solutions $\mu_4^j \in L_p(J_0; H^2_p(\mathbb{R}^n_+))$ with the estimate
\[ |\mu_4^j|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))} \leq C_j|u|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))}, \]
with some constants $C_j > 0$. Applying the divergence operator to the system (4.32) yields
\[ \text{div} \mu_4 - \mathcal{A}_0(D)(\text{div} \mu_4) = \text{div}(a \Delta u) + [\mathcal{A}_0(D), \text{div}]\mu_5, \quad \mathcal{B}(D)(\text{div} \mu_5) = [\mathcal{B}(D), \text{div}]\mu_5, \quad (4.33) \]
where $[\mathcal{A}_0(D), \text{div}]$ and $[\mathcal{B}(D), \text{div}]$ denote the commutators of $\text{div}$ and $\mathcal{A}(D)$ or $\mathcal{B}(D)$, respectively. Observe that the estimates
\[ ||[\mathcal{A}_0(D), \text{div}]\mu_5||_{L_p(J_0; L_p(\mathbb{R}^n_+))} \leq C_1|\mu_5|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))} \]
and
\[ ||[\mathcal{B}(D), \text{div}]\mu_5||_{L_p(J_0; W^{1-1/p}_{-1/p}(\mathbb{R}^{n-1}))} \leq C_2|\mu_5|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))}, \]
for some constants $C_1, C_2 > 0$ hold. Hence we may conclude that there exists a function $\mu_6 \in L_p(J_0; H^2_p(\mathbb{R}^n_+))$ such that $\mu_4 = \text{div} \mu_5 + \mu_6$ and $\mu_6$ satisfies the estimate
\[ |\mu_6|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))} \leq C|u|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))}. \]
This implies
\[ |\mu_4|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))} \leq C|u|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))}, \]
with some constant $C > 0$. Since $\mu = \mu_1 + \mu_3 + \mu_4$ this yields a constant $M > 0$ such that
\[ |\mu|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))} \leq M(|u|_{L_p(J_0; H^2_p(\mathbb{R}^n_+))} + |h_1|_{L_p(J_0; W^{1-1/p}_{-1/p}(\mathbb{R}^{n-1}))}). \]
The desired estimate for $\partial_t u$ follows from (4.34). The proof is complete. \hfill $\Box$

The continuation argument in the proof of Corollary 4.2.3 yields the following result.

**Corollary 4.3.3.** Let $a_1, c_1 \in W^1_\infty(\mathbb{R}_+^n; \mathbb{R}^n)$ and $B_1 \in W^1_\infty(\mathbb{R}_+^n; \mathbb{R}^{n \times n})$ with $\text{div} a_1(x) = \text{div} c_1(x) = 0$ for all $x \in \mathbb{R}_+^n$. Then Theorem 4.3.1 remains valid in case of variable coefficients
\[ a(x) = a^0 + a^1(x), \quad c(x) = c^0 + c^1(x) \quad \text{and} \quad B(x) = B^0 + B^1(x), \]
provided that $(\beta, a^0, c^0, B^0)$ satisfy (A), (4.10),
\[ (a^0|e_n) = (a^1|e_n) = (c^0|e_n) = (c^1|e_n) = 0, \quad x \in \partial\mathbb{R}_+^n, \]
and
\[ |a^1|_{L_\infty(\mathbb{R}_+^n; \mathbb{R}^n)} + |c^1|_{L_\infty(\mathbb{R}_+^n; \mathbb{R}^n)} + |B^1|_{L_\infty(\mathbb{R}_+^n; \mathbb{R}^{n \times n})} \leq \omega, \]
with $\omega > 0$ being sufficiently small.
4.4 Localization

In this section we prove the well-posedness of the system

\[
\begin{align*}
\partial_t u - \text{div}(a\partial_t u) &= \text{div}(B\nabla u) + f, & t > 0, & x \in \Omega, \\
\mu - c \cdot \nabla \mu &= \beta \partial_t u - \Delta u + g, & t > 0, & x \in \Omega, \\
B \nabla \mu \cdot v &= h_1, & t > 0, & x \in \Gamma, \\
\partial_u u &= h_2, & t > 0, & x \in \Gamma, \\
u(0) &= u_0, & t = 0, & x \in \Omega,
\end{align*}
\]

(4.34)

where \( \Omega \subset \mathbb{R}^n \) is a domain, with compact boundary \( \Gamma := \partial \Omega \subset C^3 \) and \( \nu = \nu(x) \) is the outer unit normal in a point \( x \in \Gamma \). We assume that the data \( a, c \) and \( B \) enjoy the regularity \( a, c \in C^1_{ub}(\Omega; \mathbb{R}^n) \) and \( B \in C^1_{ub}(\Omega; \mathbb{R}^{n \times n}) \). Suppose furthermore that the data \((\beta, a(x), c(x), B(x))\) are subject to Assumption (A) for every \( x \in \Omega \) and satisfy the conditions (4.9) and (4.10).

Let us recall some general properties of variable transformations. Suppose \( \Omega \subset \mathbb{R}^n \) is a domain with compact \( C^m \)-boundary \( \Gamma, m \in \mathbb{N} \) and let \( x_0 \in \Gamma \). Without loss of generality, we may assume that \( x_0 = 0 \) and that \( v(x_0) = [0, \ldots, 0, -1] \in \mathbb{R}^n \). This can always be achieved by a composition of a translation and a rotation in \( \mathbb{R}^n \). We remark that such affine mappings of \( \mathbb{R}^n \) onto itself leave invariant all function spaces under consideration. They also preserve ellipticity, i.e. (A) and the conditions (4.9)-(4.10). By definition of a \( C^m \)-boundary, there exists an open neighborhood \( U = U_1 \times U_2 \subset \mathbb{R}^n \) of \( x_0 \) with \( U_1 \subset \mathbb{R}^{n-1} \) and \( U_2 \subset \mathbb{R} \) as well as a function \( \rho \in C^m(U_1) \) such that

\[
\Gamma \cap U = \{x = (x',x_n) \in U : x_n = \rho(x')\},
\]

\[
\Omega \cap U = \{x = (x',x_n) \in U : x_n > \rho(x')\}.
\]

Define \( g : U \to \mathbb{R}^n \) by

\[
g_k(x) = x_k^*, \text{ if } k = 1, \ldots, n-1 \quad \text{and} \quad g_n(x) = x_n - \rho(x').
\]

(4.35)

Clearly, \( g \in C^m(\overline{U}; \mathbb{R}^n) \) is one-to-one and satisfies \( \Omega \cap U = \{x \in U : g_n(x) > 0\} \) as well as \( \Gamma \cap U = \{x \in U : g_n(x) = 0\} \). By extending \( \rho \) to a function \( \tilde{\rho} \in C^m(\mathbb{R}^n) \) with compact support and defining \( \tilde{g} \) by (4.35), with \( \rho \) replaced by \( \tilde{\rho} \), we get a \( C^m \)-diffeomorphism \( \tilde{g} \) of \( \mathbb{R}^n \) onto itself, extending \( g \) and satisfying \( \tilde{g}(x) = x \) for sufficiently large \( |x| \). Also \( \tilde{g} \) is a \( C^m \)-diffeomorphic mapping from \( \Omega_0 := \{x \in \mathbb{R}^n : x_n > \tilde{\rho}(x')\} \) onto \( \mathbb{R}_+^n \). For the Jacobian \( D\tilde{g}(x) \), one obtains

\[
D\tilde{g}(x) = \begin{bmatrix} E_{n-1} & 0 \\ -\nabla x' \cdot \tilde{\rho}(x') & 1 \end{bmatrix}, \quad x \in \mathbb{R}^n,
\]

which entails \( \det D\tilde{g}(x) = 1 \) for all \( x \in \mathbb{R}^n \) and \( D\tilde{g}(0) = E_n \). Given a function \( v \in H^m_p(\mathbb{R}_+^n) \), we define the pull back \( \Theta v \) on \( \Omega_0 \) by \( \Theta v(x) = v(\tilde{g}(x)) \). Since \( \det D\tilde{g} \equiv 1 \) and the derivatives of \( \tilde{g} \) and \( \tilde{g}^{-1} \) up to order \( m \) are bounded, the transformation formula for the Lebesgue integral shows that \( \Theta \) induces isomorphisms \( \Theta^{(p)} : H^m_p(\mathbb{R}_+^n) \to H^m_p(\Omega_0) \) for each \( p \in (1, \infty) \) and \( k \in \{0, \ldots, m\} \).

We are going to prove the following

**Theorem 4.4.1.** Let \( 1 < p < \infty \), \( J = [0,T] \) and assume that (A), (4.9) and (4.10) hold. Suppose furthermore that \( a, c \in C^1_{ub}(\Omega; \mathbb{R}^n) \) and \( B \in C^1_{ub}(\Omega; \mathbb{R}^{n \times n}) \). Then (4.34) admits a unique solution

\[
u \in H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^2_p(\Omega)) = Z^1, \quad \mu \in L_p(J; H^2_p(\Omega)) = Z^2,
\]

if and only if the data are subject to the following conditions.

(i) \( f \in L_p(J; L_p(\Omega)) = X^1 \),

(ii) \( g \in L_p(J; H^1_p(\Omega)) = X^2 \),

(iii) \( h_1 \in L_p(J; W^{1-1/p}_p(\Gamma)) = Y^1 \),
4.4. Localization

(iv) $h_2 \in W^{1-1/p}_p(J; L_p(\Gamma)) \cap L_p(J; W^{2-1/p}_p(\Gamma)) = Y^2$,
(v) $u_0 \in B^{1-2/p}_p(\Omega) = X_p$,
(vi) $\partial_t u_0 = h_2|_{t=0}$ if $p > 3/2$.

Proof. Note that due to the uniform continuity of the data, the norms of the solution operators for
the full space or half space case are uniform as well. Therefore we may cut the interval $J = [0, T]
into pieces $J_i = [i\delta, i\delta + \delta]$ for some small $\delta > 0$. We then solve the problem successively on $J_i$.
W.l.o.g. we may treat the first interval $J_0$. For this purpose we define the spaces $Z^1_k$, $X^1_k$, $Y^1_k$,
$j = 1, 2$ as the restriction of the spaces $Z^1$, $X^1$, $Y^1$ to the interval $J_0$. Furthermore we may
assume that $g = h_2 = u_0 = 0$, by solving the linear heat equation

$$\beta \partial_t u - \Delta u = -g, \quad t \in J_0, \ x \in \Omega,$n$$
$$\partial_t u = h_2, \quad t \in J_0, \ x \in \partial \Omega,$n$$
$$u(0) = u_0, \quad t = 0, \ x \in \Omega.$n

We cover $\Omega$ by finitely many open sets $U_k$, $k = 1, \ldots, N$, which are subject to the following conditions.

(i) $U_k \cap \Gamma = 0$ and $U_k = B_{r_k}(x_k)$ for all $k = 1, \ldots, N_1$;
(ii) $U_k \cap \Gamma \neq 0$ for $k = N_1 + 1, \ldots, N$.

We choose next a partition of unity $\{\varphi_k\}_{k=1}^N$ such that $\sum_{k=1}^N \varphi_k = 1$ on $\Omega$, $0 \leq \varphi_k(x) \leq 1$ and
supp $\varphi_k \subset U_j$. Note that $(u, \mu)$ is a solution of (4.34) if and only if

$$\partial_t u_{k} - \text{div}(a \partial_\nu u_{k}) = \text{div}(B \nabla \mu_{k}) + f_k + G_k(u, \mu), \quad t \in [0, \delta], \ x \in \Omega \cap U_k, \ 1 \leq k \leq N,$n$$
$$\mu_{k} - c \cdot \nabla \mu_{k} = \beta \partial_t u_{k} - \Delta u_{k} + G_k(u, \mu), \quad t \in [0, \delta], \ x \in \Omega \cap U_k, \ 1 \leq k \leq N,$n
$$B \nabla \mu_{k} \cdot \nu = h_{1k} + (B \nabla \varphi_{k} \cdot \nu)\mu, \quad t \in [0, \delta], \ x \in \Gamma \cap U_k, \ N_1 + 1 \leq k \leq N \quad (4.36)$$n$$
$$\partial_t u_{k} = u_{\nu} \varphi_{k}, \quad t \in [0, \delta], \ x \in \Gamma \cap U_k, \ N_1 + 1 \leq k \leq N,$n$$
$$u_{k}(0) = 0, \quad t = 0, \ x \in \Omega \cap U_k.$n

Here we have set $u_{k} = u_{\varphi_{k}}$, $\mu_{k} = \mu_{\varphi_{k}}$, $f_k = f_{\varphi_{k}}$ and $h_{1k} = h_{1\varphi_{k}}$. The terms $F_k(u, \mu)$ and
$G_k(u, \mu)$ are defined by

$$F_k(u, \mu) = -(a \cdot \nabla \varphi_{k}) \partial_t u - (\text{div} B \cdot \nabla \varphi_{k}) \mu - 2B \nabla \varphi_{k} \cdot \nabla \mu - (B \cdot \nabla^2 \varphi_{k}) \mu,$n$$
and

$$G_k(u, \mu) = -(c \cdot \nabla \varphi_{k}) \mu + 2\nabla u \nabla \varphi_{k} + u \Delta \varphi_{k}.$n

In case $k = 1, \ldots, N_1$ we have no boundary conditions, i.e. we only have to consider the first two
equations in (4.36). The aim is to derive an extension of the coefficients $(a(x), c(x), B(x))$ from
each ball $U_k = B_{r_k}(x_k)$ to the whole of $\mathbb{R}^n$ in order to treat these local problems with the help of
Corollary 4.2.3 for all $k = 1, \ldots, N_1$. To achieve this we have to find an extension such that
$\text{div} \bar{a}(x) = \text{div} \bar{c}(x) = 0$, $x \in \mathbb{R}^n$, for the extended coefficients $\bar{a}$ and $\bar{c}$.

We will now show how to construct such an extension. First of all note that w.l.o.g. we may
assume $x_k = 0$, $k = 1, \ldots, N_1$, after a translation in $\mathbb{R}^n$. We use the following ansatz for the
extension $\bar{a}$ of $a$.

$$\bar{a}^k(x) = \begin{cases}
a(x), & x \in B_{r_k}(0), \\
a \left(\frac{r_k^2}{x^2}\right) - 2 \left(\sum_{j=1}^n \xi_j a_j\right) \xi + R(r, \xi) \xi, & x \in \mathbb{R}^n \setminus B_{r_k}(0),
\end{cases} \quad (4.37)$$n

where $r = |x|$, $\xi = x/|x|$ and $\xi_j$, $a_j$ denote the components of $\xi$ and $a$, respectively. The scalar
valued function $R = R(r, \xi)$ will be defined later. We require $\text{div} \bar{a}^k(x) = 0$ for all $x \in \mathbb{R}^n$. Clearly
this condition is fulfilled for all \( x \in \mathbb{B}_r(0) \). So we have to compute \( \text{div} \, \tilde{a}^k(x) \) for \( x \in \mathbb{R}^n \setminus \mathbb{B}_r(0) \).

First of all we have by the chain rule

\[
\text{div} \left[ \frac{r^2 x}{r^2} \right] = \partial_i \left[ a_i \left( \frac{r^2 x}{r^2} \right) \right] = (\partial_i a_i) \left( \frac{r^2 x}{r^2} \right) \left( \frac{r^2 \delta_{ij} - 2 \frac{r^2 \xi_j \xi_i}{r^2}}{r^2} \right)
\]

\[
= \frac{r^2}{r^2} \left( \text{div} \, a \left( \frac{r^2 x}{r^2} \right) - 2 \xi_j (\partial_j a_i) \left( \frac{r^2 x}{r^2} \right) \right)
\]

\[
= -2 \frac{r^2}{r^2} \sum_{i,j} \xi_i \xi_j \partial_j a_i \left( \frac{r^2 x}{r^2} \right).
\]

These calculations imply the identity

\[
\text{div} \, \tilde{a}^k(x) = \partial_i \left[ a_i \left( \frac{r^2 x}{r^2} \right) \right] = \text{div} \, (\xi_j a_j) \xi \delta_{ij} + \text{div} (R_{\xi}).
\]

Finally we have to compute the divergence of \( R_{\xi} \), where \( R = R(r, \xi) \). We obtain

\[
\text{div} (R_{\xi}) = \partial_i (R_{\xi}) = \xi_i \partial_i R + R \cdot \left( \frac{n}{r} - \frac{\xi_i^2}{r} \right)
\]

\[
= \xi_i \partial_i R + \frac{n - 1}{r} R
\]

\[
= \xi_i \left( \xi_i \partial_i R + \frac{n - 1}{r} R \right) + \frac{n - 1}{r} R
\]

\[
= \partial_i R + \frac{n - 1}{r} R
\]

Since we require \( \text{div} \, \tilde{a}(x) = 0 \) for all \( x \in \mathbb{R}^n \), it follows that \( R = R(r, \xi) \), \( r \geq r_k \) must be a solution of the ordinary differential equation

\[
\partial_i R + \frac{(n - 1)}{r} R = 2 \frac{(n - 1)}{r} (\xi \cdot a), \quad r \geq r_k.
\]

The compatibility condition \( \tilde{a}^k(x) = a(x) \) for all \( x \in \mathbb{R}^n \) with \( |x| = r_k \) and (4.37) yield the initial condition

\[
R_k(\xi) := R(r_k, \xi) = 2(\xi \cdot a(r_k \xi)), \quad \xi = x/|x|.
\]
The function $R = R(r, \xi)$ can be explicitly computed to the result

$$
R(r, \xi) = \frac{r^{n-1}}{r^{n-1}} R_k(\xi) + \frac{2(n-1)}{r^{n-1}} \int_{r_k}^r s^{n-2}(\xi \cdot a) \, ds, \quad r \geq r_k.
$$

With the help of (4.37) we may extend the coefficients $a$ and $c$ in each ball $U_k = B_{r_k}(x_k)$, $k = 1, \ldots, N_1$ to the whole of $\mathbb{R}^n$, such that $\text{div} \tilde{a}^k(x) = \text{div} \tilde{c}^k(x) = 0$ for all $x \in \mathbb{R}^n$ and all $k = 1, \ldots, N_1$. At this point we want to emphasize that for an arbitrarily small number $\omega > 0$ we have

$$
|\tilde{a}^k(x) - a(x_k)| + |\tilde{c}^k(x) - c(x_k)| \leq \omega
$$

provided $r_k > 0$ is sufficiently small. For the coefficient matrix $B(x)$ we use the extension method from [12], i.e. we set

$$
\tilde{B}^k(x) = \begin{cases} B(x), & x \in B_{r_k}(x_k), \\ B(x_k + r_k \frac{x - x_k}{|x - x_k|^2}), & x \in \mathbb{R}^n \setminus B_{r_k}(x_k). \end{cases} \quad (4.41)
$$

This yields again $|\tilde{B}^k(x) - B(x)| \leq \omega$ for an arbitrarily small $\omega > 0$ and each $x \in \mathbb{R}^n$, provided $r_k > 0$ is sufficiently small. Hence for each chart $U_k$, $k = 1, \ldots, N_1$ we have coefficients, which fit into the setting of Corollary 4.2.3. Therefore we obtain solution operators $S^F_k \in \mathcal{B}(X^1 \times X^2; \mathcal{O}(1) \times Z^1 \times Z^2)$ of (4.36) such that

$$
\begin{bmatrix} u_k \\ \mu_k \end{bmatrix} = S^F_k \begin{bmatrix} f_k + F_k(u, \mu) \\ G_k(u, \mu) \end{bmatrix}, \quad (4.42)
$$

for each $k = 1, \ldots, N_1$.

For the remaining charts $U_k$, $k = N_1 + 1, \ldots, N$ we obtain problems in crooked half spaces with inhomogeneous Neumann boundary conditions. For the further analysis we have to understand how to treat (4.34) in such a setting. To this end we fix a point $x_0 \in \partial \Omega$ and a ball $B_{r_0}(x_0)$ with radius $r_0 > 0$ around $x_0$. After a composition of a translation and a rotation in $\mathbb{R}^n$, we may assume that $x_0 = 0$ and $\nu(x_0) = [0, \ldots, 0, -1] = e_n$. Consider a graph $\rho \in C^3(\mathbb{R}^{n-1})$, having compact support, such that

$$
\{(x', x_n) \in B_{r_0}(x_0) \subset \mathbb{R}^n : x_n = \rho(x')\} = \partial \Omega \cap B_{r_0}(x_0).
$$

Note that by decreasing the size of the charts we may assume that $|\nabla x'\rho|_{\infty}$ is as small as we like, since $\nabla x'\rho(0) = 0$. We set furthermore

$$
G = \{(x', x_n) \in \mathbb{R}^n : x_n > \rho(x')\}.
$$

We want to achieve that $\text{div} a(x) = \text{div} c(x) = 0$ for all $x \in G$ in order to apply Corollary 4.3.3, after a transformation of the crooked half space to $\mathbb{R}^n_+$. For the time being, we only know that $\text{div} a(x) = \text{div} c(x) = 0$ for all $x \in B_{r_0}(x_0) \cap G$. So we have to extend the coefficients $a$ and $c$ in a suitable way. To this end we transform the crooked boundary $\partial \Omega \cap B_{r_0}(x_0)$ to a straight line in $\mathbb{R}^{n-1}$. This will be done with the help of a transformation, which was introduced at the beginning of this section. Let $u(x', x_n) = v(g(x)) = v(x', x_n - \rho(x'))$ and $\mu(x) = \eta(g(x)) = \eta(x', x_n - \rho(x'))$, $x' \in B_{r_0}(x_0) \cap \mathbb{R}^{n-1}$. Then the differential operators $a \cdot \nabla u$ and $c \cdot \nabla \mu$ transform as follows.

$$
a(x') \cdot \nabla u(x') = a(x') \cdot (Dg(x')^T \nabla v(g(x))) = (Dg(x)a(x')) \cdot \nabla v(g(x)) = \tilde{a}(g(x)) \cdot \nabla v(g(x)),
$$

and

$$
c(x') \cdot \nabla \mu(x') = c(x') \cdot (Dg^T(x) \nabla \eta(g(x))) = (Dg(x)c(x')) \cdot \nabla \eta(g(x)) = \tilde{c}(g(x)) \cdot \nabla \eta(g(x)),
$$

with $\tilde{a}(x) := Dg(x)a(g^{-1}(x))$ and $\tilde{c}(x) = Dg(x)c(g^{-1}(x))$. Similarly we obtain

$$
\text{div}(B\nabla \mu) = \text{div}(\tilde{B}\nabla \eta),
$$
4.4. Localization

where \( B(x) := Dg(x)B(g^{-1}(x))Dg^T(x) \) and the matrix \( Dg \) is given by

\[
Dg(x) = \begin{bmatrix}
E_{n-1} & 0 \\
\nabla_x \rho(x') & 1
\end{bmatrix}, \quad x' \in B_{r_0}(x_0) \cap \mathbb{R}^{n-1},
\]

where \( E_{n-1} \) is the identity matrix in \( \mathbb{R}^{(n-1) \times (n-1)} \). The Laplace operator is transformed as follows

\[
\Delta u = \Delta v + |\nabla_x \rho|^2 \partial_y^2 v - 2 \nabla_x \rho \nabla \partial_y v - \Delta_x \rho \partial_y v,
\]

and the normal \( \nu \) at \( \partial G \) is given by

\[
\nu(x', \rho(x')) = \frac{1}{\sqrt{1 + |\nabla_x \rho|^2}} \begin{bmatrix} \nabla_x \rho' \\ -1 \end{bmatrix}.
\]

Therefore \( \sqrt{1 + |\nabla_x \rho(x')|^2} (Dg)^{-1} \nu = [0, \ldots, 0, -1]^T = e_n \), hence the transformed boundary conditions are \( B \nabla h \cdot e_n = \sqrt{1 + |\rho(x')|^2} \Theta^{-1} h_1 \) and

\[
\nabla v \cdot e_n = \frac{\Theta^{-1} h_2}{\sqrt{1 + |\nabla_x \rho|^2}} - \frac{\nabla_x \rho \cdot \nabla_x v}{1 + |\nabla_x \rho|^2}.
\]

Here \( \Theta^{-1} \) denotes the push forward operator, the inverse of the pull back operator.

Note that the set \( \Theta^{-1}(B_{r_0}(x_0) \cap \Omega) \cap \mathbb{R}^n_+ \) is not a hemisphere. Nevertheless we may choose a radius \( 0 < r_1 < r_0 \) such that

\[
B_{r_1}(x_0) \cap \mathbb{R}^n_+ \subset \Theta^{-1}(B_{r_0}(x_0) \cap \Omega).
\]

By construction, the transformed coefficients satisfy \( \text{div} \, \tilde{a}(x) = \text{div} \, \tilde{c}(x) = 0 \) for all \( x \in B_{r_1}(x_0) \cap \mathbb{R}^n_+ \)

and \( \tilde{a} \cdot e_n = \tilde{c} \cdot e_n = 0 \) for all \( x \in B_{r_1}(x_0) \cap \mathbb{R}^{n-1} \). Firstly we extend the coefficients \( \tilde{a} \) and \( \tilde{c} \) to the whole ball \( B_{r_1}(x_0) \) by even reflection in the tangential coordinates and by odd reflection w.r.t. the variable \( y \), i.e. we set

\[
\tilde{a}_{\nu}(x', y) = \begin{cases} \tilde{a}_{\nu}(x', y), & y \geq 0, \\
\tilde{a}_{\nu}(x', -y), & y \leq 0,
\end{cases}
\]

and \( \tilde{a}_g(x', y) = \tilde{a}_g(x', y) \) if \( (x', y) \in B_{r_1}(x_0) \cap \mathbb{R}^n_+ \), \( \tilde{a}_g(x', y) = -\tilde{a}_g(x', -y) \) if \( (x', y) \in B_{r_1}(x_0) \cap \mathbb{R}^{n-1} \)

and in the same way for \( \tilde{c} \). By the property \( \tilde{a} \cdot e_n = \tilde{c} \cdot e_n = 0 \) for all \( x \in B_{r_1}(x_0) \cap \mathbb{R}^{n-1} \) it holds that \( \tilde{a}, \tilde{c} \in W^{1}_2(B_{r_1}(x_0)) \)

and \( \text{div} \, \tilde{a}(x) = \text{div} \, \tilde{c}(x) = 0 \) for all \( x \in B_{r_1}(x_0) \). Now we are in a position to use the extension (4.37) in order to extend \( \tilde{a} \) and \( \tilde{c} \) to the whole of \( \mathbb{R}^n \), such that the divergence condition \( \text{div} \, \tilde{a}(x) = \text{div} \, \tilde{c}(x) = 0 \) is preserved. It is furthermore clear by the structure of (4.37) that \( \tilde{a} \cdot e_n = \tilde{c} \cdot e_n = 0 \) for all \( x \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1} \). The coefficient matrix \( B \) can be extended to a matrix \( \tilde{B} \) by the technique in [12, Proof of Theorem 8.2]. Then the condition \( \tilde{B}(x_0) \tau(x_0) \cdot e_n \) holds for all \( \tau(x_0) \in T_{x_0} \mathbb{R}^{n-1} \). We reverse the transformation to the crooked half space. This yields the following problem

\[
\begin{align*}
\partial_t u - \text{div}(a \partial_t u) &= \text{div}(B \nabla \mu) + f, \quad t > 0, \quad x \in G, \\
\mu - c \cdot \nabla \mu &= \beta \partial_t u - \Delta u + g, \quad t > 0, \quad x \in G, \\
B \nabla \mu \cdot \nu &= h_1, \quad t > 0, \quad x \in \partial G, \\
\partial_t u &= h_2, \quad t > 0, \quad x \in \partial G, \\
u(0) &= 0, \quad t = 0, \quad x \in G.
\end{align*}
\]

The coefficients \( (\beta, a, c, B) \) satisfy (A) and have the properties \( \text{div} a(x) = \text{div} c(x) = 0 \) for all \( x \in G \) and \( a(x) \cdot \nu(x) = c(x) \cdot \nu(x) = 0 \) for all \( x \in \partial G \). Furthermore the matrix \( B \) satisfies \( B(x_0) \tau(x_0) \cdot \nu(x_0) = 0 \) for \( x_0 \in \partial G \). In order to solve (4.43) we transform it again to the half space \( \mathbb{R}^n_+ \) by the procedure described above. Suppose that we already know a solution \( (u, \mu) \in \mathbb{R}^1 \times \mathbb{R}^2 \).
of (4.43). The transformation from $G$ to $\mathbb{R}_+^n$ then yields
\[
\partial_t v - \text{div}(\tilde{a}_0 \partial_t v) = \text{div}(\tilde{B} \nabla \eta) + \Theta^{-1} f, \quad t > 0, \ x \in \mathbb{R}_+^n,
\]
\[
\eta - \tilde{c} \cdot \nabla \eta = \beta \partial_t v - \Delta v + \mathcal{C}_1(x, D)v + \Theta^{-1} g, \quad t > 0, \ x \in \mathbb{R}_+^n,
\]
\[
\tilde{B} \nabla \eta \cdot e_n = \sqrt{1 + |\nabla_x \rho(x')|^2} \Theta^{-1} h_1, \quad t > 0, \ x' \in \mathbb{R}^{n-1}, \ y = 0,
\]
where the differential operators $\mathcal{C}_1(x, D)$ and $\mathcal{C}_2(x', D)$ are defined by
\[
\mathcal{C}_1(x, D)v = -|\nabla_x \rho|^2 \partial_t^2 v + 2 \nabla_x \rho \nabla \partial_y v + \Delta_x \partial_t^2 v,
\]
and
\[
\mathcal{C}_2(x', D)v = -\frac{\nabla_x \rho \cdot \nabla_x v}{1 + |\nabla_x \rho|^2}.
\]
From the extension method above it follows that
\[
|\tilde{a}(x) - a(x_0)|_{L_\infty(\mathbb{R}_+^n; \mathbb{R}^n)} + |\tilde{c}(x) - c(x_0)|_{L_\infty(\mathbb{R}_+^n; \mathbb{R}^n)} + |\tilde{B}(x) - B(x_0)|_{L_\infty(\mathbb{R}_+^n; \mathbb{R}^n)} \leq \omega,
\]
where we can choose $\omega > 0$ arbitrarily small, provided $r_0 > 0$ is sufficiently small. By Corollary 4.3.3 there exists a solution operator $S^H \in \mathcal{B}(X^1_0 \times X^2_0 \times Y^1_0 \times \mathcal{A}_D \Omega^{1,2}_\delta \times \Omega^{1,2}_\delta)$ of (4.44), i.e.
\[
[u \mu] = \Theta S^H \left[ \begin{array}{c} \Theta^{-1} f \\ \sqrt{1 + |\nabla_x \rho(x')|^2} \Theta^{-1} h_1 \\ \Theta^{-1} h_2 \\ C_1(x, D) \Theta^{-1} u \\ C_2(x', D) \Theta^{-1} u \end{array} \right].
\]
Since the solution operator is bounded and $\Theta$ is a $C^3$-diffeomorphism, we obtain the estimate
\[
|(u, \mu)|_{Z^1_1 \times Z^2_1} \leq M(|f|_{X^1_0} + |g|_{X^2_0} + |h_1|_{Y^1_0} + |h_2|_{Y^2_0} + |u|_{\mu, (J_0; H^2_p(G))} + |\nabla_x \rho|_\infty |u|_{Z^1_1}).
\]
We remind that the norm of the solution operator $S^H$ does not depend on the length $\delta > 0$ of the interval $J_0$, since we have time trace 0. This means we may again use the embeddings
\[
o Z^1_0 = o H^1 p (J_0; H^1 p(G)) \hookrightarrow L^p (J_0; H^1 p(G)) \hookrightarrow o H^{1/2} p (J_0; H^2 p(G)) \hookrightarrow L^{2p} (J_0; H^2 p(G)),
\]
to obtain $|u|_{\mu, (J_0; H^2 p(G))} \leq \delta^{1/2p} |u|_{Z^1_1}$. Since $|\nabla_x \rho|_\infty$ may be arbitrarily small, we obtain
\[
|(u, \mu)|_{Z^1_1 \times Z^2_1} \leq M(|f|_{X^1_0} + |g|_{X^2_0} + |h_1|_{Y^1_0} + |h_2|_{Y^2_0}).
\]
This means the operator $L : Z^1_1 \times Z^2_1 \to X^1_0 \times X^2_0 \times Y^1_0 \times Y^2_0$ defined by
\[
L(u, \mu) = \left[ \begin{array}{c} \partial_t u - \text{div}(a \partial_t u) - \text{div}(B \nabla \mu) \\ \mu - (c \cdot \nabla \mu) - \beta \partial_t u + \Delta u \\ (B \nabla \mu \cdot \nu) \\ \partial_t \nu \end{array} \right],
\]
is injective and has closed range, i.e. it is a semi-Fredholm operator. To show surjectivity, we apply again the homotopy argument to the set of data
\[
(\beta, a, c, B) = (1 - \tau)(\beta, 0, 0, E_n) + \tau(\beta, a, c, B), \quad \tau \in [0, 1],
\]
4.4. Localization

where \( E_n \) is the identity matrix in \( \mathbb{R}^{n \times n} \). We claim that the corresponding operator \( L_0 \) is bijective. Then \( L_1 \) is surjective, since each operator \( L_\tau \) is injective and has closed range, by the above calculations. Therefore we have to consider the system

\[
\begin{align*}
\partial_t u &= \Delta u + f, & t \in J_0, & x \in G, \\
\mu &= \beta \partial_t u - \Delta u + g, & t \in J_0, & x \in G, \\
\partial_t \mu &= h_1, & t \in J_0, & x \in \partial G, \\
\partial_\nu u &= h_2, & t \in J_0, & x \in \partial G, \\
u(0) &= 0, & t = 0, & x \in G.
\end{align*}
\]

(4.46)

Multiply the first equation by \( \beta \) and substitute \( \beta \partial_t u \) by the second equation. This yields the elliptic problem

\[
\mu - \beta \Delta \mu = \beta f + g - \Delta u, \quad \partial_t \mu = h_1.
\]

This problem admits a unique solution \( \mu \in L_p(J_0; H^2_p(G)) \) provided \( \beta f + g - \Delta u \in L_p(J_0; L_p(G)) \) and \( h_1 \in L_p(J_0; W^{1-1/p}_p(\partial G)) \). Denoting by \( \mathcal{S} \) the corresponding solution operator, we may write

\[
\mu = -\mathcal{S} \Delta u + \mathcal{S}(\beta f + g, h_1) = -\mathcal{S} \Delta u + \mu_0,
\]

with \( \mu_0 \in L_p(J_0; H^2_p(G)) \). Now we go back to (4.46) to obtain the initial boundary value problem

\[
\begin{align*}
\beta \partial_t u - \Delta u &= \mu_0 - \mathcal{S} \Delta u - g, & t \in J_0, & x \in G, \\
\partial_\nu u &= h_2, & t \in J_0, & x \in \partial G, \\
u(0) &= 0, & t = 0, & x \in G,
\end{align*}
\]

(4.47)

for the function \( u \). If \( u \in _0Z^1_\delta \) is given, then

\[
\mathcal{S} \Delta u \in \mathcal{S} H^{1/2}_p(J_0; H^2_p(G)) \cap L_p(J_0; H^3_p(G)).
\]

This means that the term \( \mathcal{S} \Delta u \) is of lower order and a Neumann series argument yields a unique solution \( u^* \in \mathcal{S}Z^1_\delta \) of (4.47). By Kato’s continuation argument it follows that the crooked half space problem (4.43) admits a unique solution

\[
(u, \mu) \in \mathcal{S} \mathcal{S} Z^1_\delta \times Z^2_\delta \text{ first on a small interval } J_0 = [0, \delta] \text{ and then also on the whole interval } J = [0, T] \text{ by a successive application of the above procedure.}
\]

We may use this result for the charts \( \psi_k \), \( k = N_1 + 1, \ldots, N \) which intersect the boundary \( \partial \Omega \), to obtain solution operators \( \mathcal{S}^H_k \in \mathcal{B}(X^1_\delta \times X^2_\delta \times Y_1 \times Y^2_\delta \times Z^1_\delta \) such that

\[
\begin{bmatrix} u_k \\ \mu_k \end{bmatrix} = \mathcal{S}^H_k \begin{bmatrix} f_k + F_k(u, \mu) \\ G_k(u, \mu) \\ h_{1k} + (B \nabla \varphi_k \cdot \nu) \mu \end{bmatrix}_{\partial_\nu \varphi_k},
\]

(4.48)

for each \( k \in \{N_1 + 1, \ldots, N\} \). Summing (4.42) and (4.48) over all charts \( \psi_k, k = 1, \ldots, N \) yields

\[
\begin{bmatrix} u \\ \mu \end{bmatrix} = \sum_{k=1}^N \mathcal{S}^F_k \begin{bmatrix} f_k + F_k(u, \mu) \\ G_k(u, \mu) \\ h_{1k} + (B \nabla \varphi_k \cdot \nu) \mu \end{bmatrix}_{\partial_\nu \varphi_k} + \sum_{k=1}^N \mathcal{S}^H_k \begin{bmatrix} f_k + F_k(u, \mu) \\ G_k(u, \mu) \\ h_{1k} + (B \nabla \varphi_k \cdot \nu) \mu \end{bmatrix}_{\partial_\nu \varphi_k},
\]

(4.49)

since \( \{\varphi_k\}_{k=1}^N \) is a partition of unity. By the boundedness of the solution operators we obtain the estimate

\[
|(u, \mu)|_{Z^1_\delta \times Z^2_\delta} \leq M \left( |f|_{X^1_\delta} + |h_1|_{Y^2_\delta} + |u|_{L_p(J_0; H^2_p(\Omega))} + |\partial_\nu u|_{L_p(J_0; L^p(\partial \Omega))} + |\mu|_{L_p(J_0; H^3_p(\Omega))} \right),
\]

(4.50)

for some constant \( M > 0 \). The term \( |u|_{L_p(J_0; H^2_p(\Omega))} \) may be estimated by \( \delta^{1/2} C |u|_{Z^2_\delta} \) with some constant \( C > 0 \), while for the last two terms in (4.50) we need an estimate like that of Proposition 4.3.2 but here for the domain \( \Omega \). The arguments for a general domain \( \Omega \subset \mathbb{R}^n \) are similar to those
in the proof of Proposition 4.3.2. Indeed, it suffices to show that the $L_p$-realization $A_0$ of the differential operator

$$A_0(D)w = c \cdot \nabla w + a \cdot \nabla w - \text{div}(a(c \cdot \nabla w)) + \text{div} (\beta B \nabla w)$$

with domain

$$D(A_0) = \{ v \in L_p(J_0; H^1_p(\Omega)) : B \nabla v \cdot \nu = 0 \text{ on } \partial \Omega \},$$

is dissipative in $L_p(J_0; L_p(\Omega))$. But due to the assumptions $\text{div}(a(x)) = \text{div}(c(x)) = 0, \ x \in \Omega$, and $a(x) \cdot \nu(x) = c(x) \cdot \nu(x) = 0, \ x \in \partial \Omega$, this follows immediately from the proof of Proposition 4.2.2. Hence, choosing $\delta > 0$ small enough we therefore obtain

$$\|(u, \mu)|_{Z^1_0 \times Z^2_0} \leq M(\|f\|_{X^1_0} + |g|_{X^2_0} + |h_1|_{Y^1_0} + |h_2|_{Y^2_0} + |u_0|_{X_0}),$$

(4.51)

for the solution $(u, \mu) \in Z^1_0 \times Z^2_0$ of (4.34). Now we may again employ the continuation argument of Kato to see that the solution operator to (4.34) is bijective. This can be done as in the case of a crooked half space. The proof is complete.

4.5 Local Well-Posedness

We are going to solve the semilinear problem

$$\partial_t \psi - \text{div}(a \partial_t \psi) = \text{div}(B \nabla \mu) + f, \quad t > 0, \ x \in \Omega,$$

$$\mu - c \cdot \nabla \mu = \beta \partial_t \psi - \Delta \psi + \Phi'(\psi) + g, \quad t > 0, \ x \in \Omega,$$

$$B \nabla \psi \cdot \nu = h_1, \quad t > 0, \ x \in \Gamma,$$

$$\partial_t \psi = h_2, \quad t > 0, \ x \in \Gamma,$$

$$\psi(0) = \psi_0, \quad t = 0, \ x \in \Omega,$$

(4.52)

where the data $\beta, a, c, B$ are subject to Assumption (A), (4.9) and (4.10) and let $a, c \in C^{1}_a(\Omega; \mathbb{R}^n)$, $B \in C^{1}_a(\Omega; \mathbb{R}^{n \times n})$. To this end let $f \in X^1, \ g \in X^2, \ h_j \in Y_j, \ j = 1, 2$ and $\psi_0 \in X_p$ be given such that the compatibility condition $\partial_t \psi_0 = h_2|_{t=0}$ if $p > 3/2$ is satisfied. Applying Theorem 4.4.1 we may define a pair of functions $(u^*, \nu^*) \in Z^1 \times Z^2$ as the unique solution of

$$u^*_t - \text{div}(a u^*_t) = \text{div}(B \nabla \nu^*) + f, \quad t > 0, \ x \in \Omega,$$

$$\nu^* - c \cdot \nabla \nu^* = \beta u^*_t - \Delta u^* + g, \quad t > 0, \ x \in \Omega,$$

$$B \nabla \nu^* \cdot \nu = h_1, \quad t > 0, \ x \in \Gamma,$$

$$\partial_t u^* = h_2, \quad t > 0, \ x \in \Gamma,$$

$$u^*(0) = \psi_0, \quad t = 0, \ x \in \Omega.$$

(4.53)

We set

$$E_1 = Z^1(T) \times Z^2(T), \quad 0 E_1 = \{(u, v) \in E_1 : u|_{t=0} = 0\},$$

$$E_0 = X^1(T) \times X^2(T) \times Y_1(T) \times Y_2(T), \quad 0 E_0 = \{ (f, g, h_1, h_2) \in E_0 : h_2|_{t=0} = 0\}$$

and denote by $| \cdot |_1$ and $| \cdot |_0$ the canonical norms in $E_1$ and $E_0$, respectively. Following the lines of Chapters 2 & 3 we define a linear operator $L : 0 E_1 \rightarrow 0 E_0$ by

$$L(u, v) = \begin{bmatrix}
\partial_t u - \text{div}(a \partial_t u) - \text{div}(B \nabla v) \\
\nu - c \cdot \nabla \nu - \beta \partial_t u + \Delta u \\
B \nabla \nu \cdot \nu \\
\partial_t u
\end{bmatrix}$$

and a nonlinear function $G : 0 E_1 \times E_1 \rightarrow 0 E_0$ by

$$G((u, v), (u^*, \nu^*)) = \begin{bmatrix}
0 \\
\Phi'(u + u^*) \\
0 \\
0
\end{bmatrix}$$
Again we consider $L$ as an operator from $\mathcal{D}(\mathcal{L}_1)$ to $\mathcal{D}(\mathcal{L}_0)$. Hence Theorem 4.4.1 yields that $L$ is a bounded isomorphism and by the open mapping theorem $L$ is invertible with bounded inverse $L^{-1}$. It is easily seen that $(\psi, \mu) := (u + u^*, v + v^*)$ is a solution of (4.52) if and only if
\[ L(u, v) = G((u, v), (u^*, v^*)) \] or equivalently
\[ (u, v) = L^{-1}G((u, v), (u^*, v^*)). \]

Consider a ball $B_R \subset \mathbb{E}_1$ where $R \in (0, 1]$ will be fixed later. To apply the contraction mapping principle we furthermore define a nonlinear operator by $T(u, v) := L^{-1}G((u, v), (u^*, v^*))$. As in Chapters 2 & 3 we have to show that $T(B_R) \subset B_R$ and that there exists a constant $\kappa < 1$ such that the contraction inequality
\[ |T(u, v) - T(\bar{u}, \bar{v})|_1 \leq \kappa |(u, v) - (\bar{u}, \bar{v})|_1 \] holds for all $(u, v), (\bar{u}, \bar{v}) \in B_R$. We first care about the contraction mapping property. By Hölder’s inequality and with the assumption $\Phi \in C^{3-}(\mathbb{R})$ we obtain
\[ |T(u, v) - T(\bar{u}, \bar{v})|_1 \leq |L^{-1}||G((u, v), (u^*, v^*)) - G((\bar{u}, \bar{v}), (u^*, v^*))|_0 \]
\[ \leq M|\Phi'(u + u^*) - \Phi'(\bar{u} + u^*)|_{X_2(T)} \]
\[ \leq M|\Phi'(u + u^*) - \Phi'(\bar{u} + u^*)|_{p,p} \]
\[ + |\nabla(\Phi'(u + u^*) - \Phi'(\bar{u} + u^*))|_{p,p} \]
\[ \leq M\left( |u - \bar{u}|_{p,p} + |\nabla(u + u^*)|_{r,p,r} \right) \Phi''(u + u^*) - \Phi''(\bar{u} + u^*)|_{r,r'} \]
\[ \leq MT^{1/r'}\left( |u - \bar{u}|_{r,p,r} + |\nabla u - \nabla \bar{u}|_{r,p,r} \right) \]
\[ \leq \kappa(T)|((u, v) - (\bar{u}, \bar{v})|_1, \]
where $\kappa = \kappa(T)$ is a function with the property that $\kappa(T) \to 0$ as $T \to 0$, and a constant $M > 0$ which does not depend on $T$, since time traces are equal to 0 at $t = 0$, whenever they exist. Here we made use of the embedding
\[ H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^3_p(\Omega)) \hookrightarrow C(J \times \Omega), \]
provided $p > (n + 2)/3$. Furthermore, since in the above calculation we may chose $r > 1$ arbitrarily close to 1, it holds that
\[ H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^3_p(\Omega)) \hookrightarrow L_{rp}(J; H^1_{rp}(\Omega)). \]

Thus, if $T$ is sufficiently small we obtain (4.54). The self mapping property can be shown in a similar way. The above computation yields
\[ |T(u, v)|_1 \leq |T(u, v) - T(0, 0)|_1 + |T(0, 0)|_1 \]
\[ \leq \kappa(T)|u, v|_1 + M|G(0, 0, (u^*, v^*))|_0 \]
\[ \leq \kappa(T)|u, v|_1 + M|\Phi'(u^*)|_{X_2(T)} \]
\[ \leq \kappa(T)R + M|\Phi'(u^*)|_{X_2(T)}. \]

Since $\Phi'(u^*)$ is a fixed function in $X_2(T)$ it follows that $|\Phi'(u^*)|_{X_2(T)} \to 0$ as $T \to 0$, whence $T(B_R) \subset B_R$, provided that $T > 0$ is small enough. The contraction mapping principle yields a unique fixed point $(\bar{u}, \bar{v}) \in \mathcal{D}(\mathcal{L}_1)$ or equivalently $(\psi, \mu) := (u + u^*, \bar{v} + v^*) \in \mathbb{E}_1$ is the unique local solution of (4.52). Therefore we have the following result.

**Theorem 4.5.1.** Let $1 < p < \infty$ and $p > (n + 2)/3$. Assume furthermore that $\Phi \in C^{3-}(\mathbb{R})$ and let (A) as well as (4.9),(4.10) be satisfied. Suppose that $a, c \in C^1_{ub}(\Omega; \mathbb{R}^{n \times n})$ and $B \in C^1_{ub}(\Omega; \mathbb{R}^{n \times n})$. Then there exists an interval $J = [0, T] \subset [0, T_0]$ and a unique solution $(\psi, \mu)$ of (4.52) on $J$, with
\[ \psi \in H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^3_p(\Omega)) = Z^1(T) \]
4.6 Global Well-Posedness

Throughout this section, we assume that $p \geq 2$ and $n \leq 3$. Furthermore, we will need the following assumption.

**H** There exists a constant $\varepsilon > 0$ such that

$$
\beta z_0^2 + (a + c|z_1|)z_0 + (Bz_1|z_1|) \geq \varepsilon (z_0^2 + |z_1|^2),
$$

for all $(z_0, z_1) \in \mathbb{R} \times \mathbb{R}^n$.

This condition is crucial in order to obtain some energy estimates, which will be used in the proof of global well-posedness. We will show in the Appendix, that (H) already implies (A). Assume furthermore that the data $(\beta, a, c, B)$ satisfy (4.9), (4.10) and let $a, c \in C^1_{\text{ub}}(\Omega; \mathbb{R}^n)$ and $B \in C^1_{\text{ub}}(\Omega; \mathbb{R}^{n \times n})$.

A successive application of Theorem 4.5.1 yields a maximal interval of existence $T_{\text{max}} = [0, T_{\text{max}})$ for the solution $(\psi, \vartheta)$ of (4.52). In order to prove the global existence of $\psi$ on $\mathbb{R}_+$, we have to verify that $|\psi|_{L^1(T)}$ is uniformly bounded for all $T \in I$ and all compact intervals $I \subset \mathbb{R}_+$. The embedding

$$
L^1(T) \hookrightarrow C([0, T]; B_{pp}^{3-2/p}(\Omega))
$$

then yields that the limit $\lim_{t \to T_{\text{max}}} |\psi(t)|_{X_p}$ exists, which means that we can continue the solution $\psi$ beyond $T_{\text{max}}$. Then it follows from the equations that $\mu$ exists globally, too. In other words this means that $T_{\text{max}} = +\infty$. Let $J_0 = [0, T_0]$ and let $T \in J_0$. The open mapping theorem yields the estimate

$$
|\psi|_{L^1(T)} + |\mu|_{L^1(T)} \leq M(T_0) \left( |\Phi'(\psi)|_{X^2(T)} + |f|_{X^2(T_0)} + |g|_{X^2(T_0)} + |h_1|_{Y^1(T_0)} + |h_2|_{Y^2(T_0)} + |\psi_0|_{X_p} \right) \quad (4.57)
$$

for the local solution $(\psi, \mu) \in \mathcal{E}_1$ of (4.52). First of all we will derive an a priori estimate for $\psi$. To do so we multiply (4.52)$_1$ by $\mu$, (4.52)$_2$ by $-\partial_t \psi$ and integrate by parts to obtain

$$
\int_{\Omega} \left( \partial_t \psi \mu + (B \nabla \mu \nabla \psi) + (a \nabla \mu) \partial_t \psi \right) \, dx = \int_{\Omega} \mu f \, dx + \int_{\Gamma} \mu h_1 \, d\Gamma \quad (4.58)
$$
and
\[
\int_\Omega \left( -\partial_t \psi \mu + (c|\nabla \mu| \partial_t \psi + \beta |\partial_t \psi|^2 + \frac{1}{2} \frac{\partial}{\partial t} |\nabla \psi|^2 + \frac{\partial}{\partial t} \Phi(\psi) \right) \, dx = \int_\Gamma \partial_t \psi h_2 \, d\Gamma - \int_\Omega \partial_t \psi g \, dx. \tag{4.59}
\]

Adding (4.58) and (4.59) yields the equation
\[
\frac{d}{dt} \left( \frac{1}{2} |\nabla \psi|^2 + \int_\Omega \Phi(\psi) \, dx \right) + \beta |\partial_t \psi|^2 + (a + c|\partial_t \psi \nabla \mu|_2) + (B|\nabla \mu| \nabla \mu)_2
\]
\[
= \int_\Omega \mu f \, dx + \int_\Gamma \mu h_1 \, d\Gamma + \int_\Omega \partial_t \psi h_2 \, d\Gamma - \int_\Omega \partial_t \psi g \, dx. \tag{4.60}
\]

From Assumption (H) with \( z_0 = \partial_t \psi \) and \( z_1 = \nabla \mu \) it follows that
\[
\beta |\partial_t \psi|^2 + (a + c|\partial_t \psi \nabla \mu|_2) + (B|\nabla \mu| \nabla \mu)_2 \geq \varepsilon (|\partial_t \psi|_2^2 + |\nabla \mu|_2^2).
\]

For the first and the second integral in (4.60) we apply Hölder’s inequality as well as the Poincaré-Wirtinger inequality to obtain
\[
\int_\Omega \mu f \, dx \leq C|f|_{2} \left( |\nabla \mu|_2 + |\int_\Omega \mu \, dx| \right) \quad \text{and} \quad \int_\Gamma \mu h_1 \, d\Gamma \leq C|h_1|_{2, \Gamma} \left( |\nabla \mu|_2 + |\int_\Omega \mu \, dx| \right).
\]

The integral \( \int_\Omega \mu \, dx \) can be computed in the following way. Assuming that \( \text{div} \, c = 0 \) in \( \Omega \) and \( (c |\nu|) = 0 \) as well as \( (a |\nu|) = 0 \) on \( \Gamma \) we have
\[
\int_\Omega (c|\nabla \mu|) \, dx = \int_\Gamma (c|\nu|) \, d\Gamma - \int_\Omega \text{div} \, c \, dx = 0,
\]

hence it follows from (4.52) and the boundary conditions that
\[
\int_\Omega \mu \, dx = \beta \int_\Omega \partial_t \psi \, dx + \int_\Omega \Phi'(\psi) \, dx + \int_\Omega g \, dx
\]
\[
= \int_\Omega \Phi'(\psi) \, dx + \int_\Omega g \, dx + \beta \left( \int_\Omega f \, dx + \int_\Gamma h_1 \, d\Gamma \right).
\]

With the additional assumption
\[
|\Phi'(s)| \leq (c_1 \Phi(s) + c_2 s^2 + c_3)\theta, \quad \text{for all } s \in \mathbb{R}, \tag{4.61}
\]

with some constants \( c_i > 0, \theta \in (0, 1) \), we obtain
\[
|\int_\Omega \mu \, dx| \leq \int_\Omega (c_1 \Phi(\psi) + c_2 |\psi|^2 + c_3)\theta \, dx + c(|g|_1 + |h_1|_{1, \Gamma} + |f|_1).
\]

By the last estimate, Young’s inequality and the Poincaré inequality it holds that
\[
\int_\Omega \mu f \, dx + \int_\Gamma \mu h_1 \, d\Gamma \leq C(\delta) \left( |\nabla \psi|^2 + \int_\Omega \Phi(\psi) \, dx + |f|^2_{\theta, 2} + |h_1|^2_{\theta, \Gamma} + |g|^2_{\theta, 2} + 1 \right) + \delta |\nabla \mu|_2^2. \tag{4.62}
\]

where \( q := \max\{2, \frac{1}{1-\theta} \} \) and \( \delta > 0 \) may be arbitrarily small. For the term \( \int_\Omega \partial_t \psi \, dx \) in (4.60) we apply Young’s inequality one more time to obtain
\[
\int_\Omega \partial_t \psi g \, dx \leq \delta |\partial_t \psi|^2 + C(\delta)|g|^2_{\theta, 2}. \tag{4.63}
\]

Integrating (4.60) with respect to \( t \) and choosing \( \delta > 0 \) small enough, we obtain together with (4.62) and (4.63) the estimate
\[
\frac{1}{2} |\nabla \psi(t)|^2_2 + \int_\Omega \Phi(\psi(t)) \, dx + C_1(|\partial_t \psi|_{2, 2}^2 + |\nabla \mu|_{2, 2}^2)
\]
\[
\leq C_2 \left( \int_0^t \left( \frac{1}{2} |\nabla \psi(\tau)|^2_2 + \Phi(\psi(\tau)) \right) \, d\tau + |f|^2_{\theta, 2} + |h_1|^2_{\theta, \Gamma} + |g|^2_{\theta, 2} + 1 \right)
\]
\[
+ \int_0^t \int_\Gamma \partial_t \psi h_2 \, d\Gamma \, d\tau. \tag{4.64}
\]
In order to treat the last double integral, we have to assume more regularity for the function \( h_2 \).
To be precise, we assume that
\[
h_2 \in H^1_\delta(J; L_\infty(\Gamma)) \cap L_p(J; W^{2-1/p}_\delta(\Gamma)) \hookrightarrow C(J; L_p(\Gamma)).
\]
Due to this fact, we may integrate the last term in (4.64) by parts to the result
\[
\int_0^t \int_\Gamma \partial_t h_2 \ d\Gamma \ d\tau = \int_\Gamma \psi(t) h_2(t) \ d\Gamma - \int_\Gamma \psi_0 |h_2|_{t=0} - \int_0^t \int_\Gamma \partial_t h_2 \ d\Gamma \ d\tau,
\]
(4.65)
where we also made use of Fubini’s theorem. For the first term we use Young’s inequality, the
embedding \( H^1_\delta(\Omega) \hookrightarrow L_2(\Gamma) \) and the fact that
\[
\int_\Omega \psi(t) \ dx = \int_\Omega \psi_0 \ dx + \int_0^t \int_\Omega f \ dx \ d\tau + \int_0^t \int_\Gamma h_1 \ d\Gamma \ d\tau.
\]
This yields
\[
\int_\Gamma \psi(t) h_2(t) \ d\Gamma \leq \delta |\psi(t)|_{H^1_\delta(\Gamma)}^2 + C_0 |h_2(t)|_{L_2(\Gamma)}^2,
\]
\[
\leq \eta C |\nabla \psi(t)|_{L_2(\Gamma)}^2 + C(\eta) \left( |\psi|_{L_\infty(\Omega)}^2 + |f|_{L_1(\Gamma)} + |h_1|_{L_2(\Gamma)} + |\psi_0|_{L_2(\Gamma)} \right).
\]
Next, by Theorem 4.5.1 (vi) it holds that \( h_2|_{t=0} = \partial_x \psi_0 \in B^{3-3/p}_{pp}(\Gamma) \hookrightarrow L_2(\Gamma) \), if \( p > 3/2 \) and by
trace theory, we obtain
\[
B^{3-3/p}_{pp}(\Omega) \hookrightarrow B^{3-3/p}_{pp}(\Gamma) \hookrightarrow L_2(\Gamma).
\]
These embeddings ensure that the integral \( \int_\Gamma \psi_0 |h_2|_{t=0} \ d\Gamma \) converges. Finally, concerning the last
term in (4.65) we use Young’s inequality one more time to the result
\[
\int_0^t \int_\Gamma \partial_t h_2 \ d\Gamma \ d\tau \leq \frac{1}{2} \int_0^t |\psi(\tau)|_{H^1_\delta(\Omega)}^2 \ d\tau + \frac{1}{2} |\partial_t h_2|_{L_2(\Gamma)}^2,
\]
\[
\leq C \int_0^t |\nabla \psi(\tau)|_{L_2(\Gamma)}^2 \ d\tau + C(T_0, f, h_1, h_2, \partial_t h_2, \psi_0),
\]
where we used again (4.66). Set
\[
E(u) = \frac{1}{2} |\nabla u|_{L_2(\Gamma)}^2 + \int_\Omega \Phi(u) \ dx, \quad u \in H^1_\delta(\Omega).
\]
Then by the above estimates there exist some constants \( C_0 > 0 \) such that
\[
E(\psi(t)) + C_1 (|\partial_t \psi|_{L_2(\Gamma)}^2 + |\nabla |_{L_2(\Gamma)}^2) \leq C_2 \int_0^T \int \psi(t) \ d\tau + C_0 (T_0, f, g, h_1, h_2, \partial_t h_2, \psi_0),
\]
provided that \( \delta > 0 \) is sufficiently small. Assume that \( \Phi \) satisfies the additional condition
\[
\Phi(s) \geq -\frac{\eta}{2} s^2 - c_0, \quad s \in \mathbb{R},
\]
(4.67)
where \( c_0 > 0 \) and \( \eta < \lambda_1 \), with \( \lambda_1 \) being the first nontrivial eigenvalue of the negative Neumann
Laplacian. With the help of (4.67) it follows that \( E(u) \) is bounded from below for all \( u \in H^1_\delta(\Omega) \),
and hence we may apply Gronwall’s lemma to the result that \( E(\psi(\cdot)) \) is bounded on \( J_{\max} = [0, T_{\max}) \).
Applying (4.67) one more time and using the fact that \( |\int_\Omega \psi(t, x) \ dx| \) is bounded it holds that
\[
\psi \in L_\infty(J_{\max}; H^1_\delta(\Omega)).
\]
Applying the same arguments as in the proof of Lemma 3.4.1, we obtain an inequality of the form
\[
|\Phi'(\psi)|_{L^\infty(J_{\max}; H^1_\delta(\Omega))} \leq C(1 + |\psi|_{L^\infty(J_{\max}; H^1_\delta(\Omega))}^n),
\]
where we used again (4.66).
for some constants $C > 0$, $m > 0$ and $\delta \in (0,1)$, provided that the potential $\Phi \in C^3(\mathbb{R})$ satisfies the growth condition

$$|\Phi'''(s)| \leq c_0(1 + |s|^\gamma), \quad s \in \mathbb{R},$$

(4.68)

with $\gamma < 3$ in case $n = 3$ and some constant $c_0 > 0$. Hence it follows from (4.57) that $|\psi|_{Z^1(T)}$ is bounded with respect to $T \in [0,T_0]$ and this means that we may continue the solution beyond $T_{\text{max}}$. Thus we obtain the following result on global well-posedness.

**Theorem 4.6.1.** Let $p \geq 2$, $n \leq 3$, $q = \max\{2, \frac{1}{1-\gamma}\}$, with $\gamma$ from (4.61), and let Hypotheses (H) as well as (4.9),(4.10) hold. Suppose that $a, c \in C^1(\overline{\Omega}; \mathbb{R}^n)$ and $B \in C^1(\overline{\Omega}; \mathbb{R}^{n \times n})$. Assume furthermore that $\Phi$ satisfies (4.61), (4.67) and (4.68). Then there exists a unique global solution $(\psi, \mu)$ of (4.52) on $J_0 = [0,T_0]$, with

$$\psi \in H^1_p(J_0; H^1_p(\Omega)) \cap L_p(J_0; H^3_p(\Omega))$$

and

$$\mu \in L_p(J_0; H^2_p(\Omega)),$$

provided that the data are subject to the following conditions.

(i) $f \in L_p(J_0; \mu(\Omega)) \cap L_q(J_0; L_2(\Omega))$,

(ii) $g \in L_p(J_0; H^1_p(\Omega))$,

(iii) $h_1 \in L_p(J_0; W^{1-1/p}_p(\Gamma)) \cap L_q(J_0; L_2(\Gamma))$,

(iv) $h_2 \in H^1_p(J_0; L_p(\Gamma)) \cap L_p(J_0; W^{2-1/p}_p(\Gamma))$,

(v) $\psi_0 \in B^{3-2/p}_p(\Omega)$,

(vi) $\partial_t \psi_0 = h_2|_{t=0}$, if $p > 3/2$.

The solution depends continuously on the given data and if the data are independent of $t$, the map

$$\psi_0 \mapsto \psi(t), \quad t \in \mathbb{R}_+,$$

defines a global semiflow on the natural phase manifold $\mathcal{M}_p$.

### 4.7 Asymptotic Behavior

In this last section we will give a qualitative analysis of global solutions of the Cahn-Hilliard-Gurtin system

$$\partial_t \psi - \text{div}(a \partial_t \psi) = \text{div}(B \nabla \mu), \quad t > 0, \quad x \in \Omega,$$

$$\mu - c \cdot \nabla \mu = \beta \partial_t \psi - \Delta \psi + \Phi'(\psi), \quad t > 0, \quad x \in \Omega,$$

$$B \nabla \mu \cdot \nu = 0, \quad t > 0, \quad x \in \Gamma,$$

$$\partial_t \psi = 0, \quad t > 0, \quad x \in \Gamma,$$

$$\psi(0) = \psi_0, \quad t = 0, \quad x \in \Omega.$$  

(4.69)

To be more precise we will show that each trajectory converges to a stationary point, i.e. to a solution of the corresponding stationary system. The so called *Lojasiewicz-Simon inequality* will play an important role in the proof of this assertion. Assume that the data $(\beta, a, c, B)$ satisfy (4.9), (4.10) and (H). Suppose furthermore that $a, c \in C^1(\overline{\Omega}; \mathbb{R}^n)$ and $B \in C^1(\overline{\Omega}; \mathbb{R}^{n \times n})$. Let $\psi_0 \in \mathcal{M}_2$ and let $(\psi, \mu)$ be the unique global solution of (4.69). We recall from Section 4.6 the energy functional

$$E(u) = \frac{1}{2} |\nabla u|^2 + \int_{\Omega} \Phi(u) \, dx,$$

defined on the energy space

$$V := \{u \in H^1_3(\Omega) : \int_{\Omega} u \, dx = 0\}.$$
Note that due to (4.69), and the boundary condition (4.69), we obtain \( \int_{\Omega} \psi \, dx = \int_{\Omega} \psi_0 \, dx \), since \((a|\nu) = 0\) on \(\Gamma\). If we perform a shift of \(\psi\) by means of \(\tilde{\psi} = \psi - c\), where \(c := \int_{\Omega} \psi_0 \, dx\), it follows that \(\tilde{\psi}\) is again a solution of (4.69), if we replace the physical potential \(\Phi\) by \(\tilde{\Phi}(s) = \Phi(s + c)\) and additionally it holds that \(\int_{\Omega} \tilde{\psi} \, dx = 0\). It follows from (4.60) that in the homogeneous case \(E(\psi(\cdot))\) satisfies the equation

\[
\frac{d}{dt} E(\psi(t)) + \beta |\partial_t \psi(t)|^2 + (a + c) |\partial_t \psi(t) \nabla \mu(t)|^2 + (B \nabla \mu(t) | \nabla \mu(t)|)^2 = 0,
\]

for all \(t \in \mathbb{R}_+\). Making again use of Hypothesis (H) we obtain the inequality

\[
\frac{d}{dt} E(\psi(t)) + \varepsilon (|\partial_t \psi(t)|^2 + |\nabla \mu(t)|^2) \leq 0,
\]

which holds for all \(t \in \mathbb{R}_+\). Integrating with respect to \(t\) and making use of (4.67) as well as of the Poincaré inequality we obtain the a priori estimates

\[
\psi \in L_\infty(\mathbb{R}_+; H^1_0(\Omega)) \quad \text{and} \quad \partial_t \psi, |\nabla \mu| \in L_2(\mathbb{R}_+ \times \Omega).
\]

**Proposition 4.7.1.** The orbit \(\{\psi(t)\}_{t \in \mathbb{R}_+}\) is relatively compact in \(V\).

**Proof.** We rewrite equation (4.69), as follows

\[
\beta \partial_t \psi - \Delta \psi + \psi = \mu - \overline{\mu} - (c(x) |\nabla \mu|) + \overline{\mu} + \psi - \Phi'(\psi),
\]

where \(\overline{\mu} = \frac{1}{|\Omega|} \int_{\Omega} \Phi'(\psi) \, dx\). By the energy estimates above and the Poincaré-Wirtinger inequality it follows that

\[
f := \mu - \overline{\mu} + (c |\nabla \mu|) \in L_2(\mathbb{R}_+; L_2(\Omega)).
\]

Furthermore we have

\[
g := \overline{\mu} + \psi - \Phi'(\psi) \in L_\infty(\mathbb{R}_+; L_q(\Omega)),
\]

where \(q = 6/(\gamma + 2)\) is determined by the growth condition (4.68) on \(\Phi\). The operator \(A := -\Delta + I\) with domain

\[
D(A) = \{u \in H^2_p(\Omega) : \partial_n u = 0 \text{ on } \Gamma\}
\]

generates an exponentially stable, analytic \(C_0\)-semigroup \(\{T(t)\}_{t \in \mathbb{R}_+}\) in \(L_p(\Omega)\). Therefore

\[
T(\cdot) * f \in H^1_2(\mathbb{R}_+; L_2(\Omega)) \cap L_2(\mathbb{R}_+; H^2_0(\Omega)) \hookrightarrow C_0(\mathbb{R}_+; H^1_2(\Omega)).
\]

For the function \(g\) we apply elementary semigroup theory to obtain

\[
T(\cdot) * g \in C_0(\mathbb{R}_+; H^1_2(\Omega)),
\]

for each \(s \in (0, 2)\). The space \(H^s_0(\Omega)\) embeds compactly into \(H^1_2(\Omega)\), if \(s\) is chosen close enough to 2. This completes the proof of relative compactness, since \(\psi_0 \in H^2_0(\Omega)\).

\[
\square
\]

The following proposition provides some crucial properties of the \(\omega\)-limit set

\[
\omega(\psi) = \{\varphi \in V : \exists (t_n) \nearrow \infty, \text{ s.t. } \psi(t_n) \to \varphi \text{ in } V\}.
\]

**Proposition 4.7.2.** Suppose that \((\psi, \mu)\) is a global solution of (4.69) and let \(\Phi\) satisfy Hypotheses (4.67) and (4.68). Then the following statements hold.

(i) The mapping \(t \mapsto E(\psi(t))\) is nonincreasing and the limit \(\lim_{t \to \infty} E(\psi(t)) := E_\infty \in \mathbb{R}\) exists.

(ii) The \(\omega\)-limit set \(\omega(\psi) \subset V\) is nonempty, connected, compact and \(E\) is constant on \(\omega(\psi)\).
(iii) Every $\psi_\infty \in \omega(\psi)$ is a strong solution (in the sense of $L_2$) of the stationary problem

$$-\Delta \psi_\infty + \Phi' (\psi_\infty) = \mu_\infty, \quad x \in \Omega,$$

$$\partial_\nu \psi_\infty = 0, \quad x \in \Gamma,$$  

(4.71)

where $\mu_\infty = \frac{1}{|\Omega|} \int_\Omega \Phi' (\psi_\infty) \, dx = \text{const.}$

(iv) Each $\psi_\infty \in \omega(\psi)$ is a critical point of $E$, i.e. $E'(\psi_\infty) = 0$ in $V^*$, where $V^*$ is the topological dual space of $V$.

Proof. From the inequality (4.70) it follows that $E(\psi(\cdot))$ is nonincreasing with respect to $t$. Furthermore by (4.67) it follows that $E(u)$ is bounded from below for all $u \in V$. This proves (i).

Assertion (ii) follows easily from well-known facts in the theory of dynamical systems.

Let $\psi_\infty \in \omega(\psi)$. Then there exists a sequence $(t_n) \not\to +\infty$ such that $\psi(t_n) \to \psi_\infty$ in $V$ as $n \to \infty$. Since $\partial_t \psi \in L_2(\mathbb{R}_+ \times \Omega)$ it follows that $\psi(t_n + s) \to \psi_\infty$ in $L_2(\Omega)$ for all $s \in [0, 1]$ and by relative compactness also in $V$. This can be seen as in Chapter 2. Integrating (4.70) from $t_n$ to $t_n + 1$ we obtain

$$E(\psi(t_n + 1)) - E(\psi(t_n)) + \varepsilon \int_0^1 \int_\Omega (|\nabla \mu(t_n + s, x)|^2 + |\partial_\nu \psi(t_n + s, x)|^2) \, dx \, ds \leq 0.$$ 

Letting $t_n \to +\infty$ yields

$$|\nabla \mu(t_n + s, \cdot) , \partial_\nu \psi(t_n + s, \cdot)| \to 0 \quad \text{in} \quad L_2([0, 1] \times \Omega).$$

This in turn yields a subsequence $(t_{n_k})$ such that $|\nabla \mu(t_{n_k} + s), \partial_\nu \psi(t_{n_k} + s) \to 0$ in $L_2(\Omega)$ for a.e. $s \in [0, 1]$. We fix such an $s$, say $s^* \in [0, 1]$. The Poincaré-Wirtinger inequality implies that

$$\mu(t_{n_k} + s^*) - \mu(t_{n_k} + s^*) |_{L_2} \leq C_p \left( |\nabla \mu(t_{n_k} + s^*) - \nabla \mu(t_{n_k} + s^*)|_{L_2} + \int_\Omega |\Phi'(\psi(t_{n_k} + s^*)) - \Phi'(\psi(t_{n_k} + s^*))| \, dx \right),$$

since $\int_\Omega \mu \, dx = \int_\Omega \Phi'(\psi) \, dx$. Letting $k \to \infty$ and making use of (4.68) it follows that the solution $\mu(t_{n_k} + s^*)$ is a Cauchy sequence in $L_2(\Omega)$, hence it admits a limit, which we denote by $\mu_\infty$. Since the gradient is a closed operator in $L_2(\Omega; \mathbb{R}^n)$ it holds that $\mu_\infty \in H^1_2(\Omega)$ and $\nabla \mu_\infty = 0$. Thus $\mu_\infty = \text{const.}$ and we have the identity $\mu_\infty = \frac{1}{|\Omega|} \int_\Omega \Phi'(\psi_\infty) \, dx$. Finally we multiply (4.69) by a function $\varphi \in V$ in $L_2(\Omega)$ to the result

$$\beta (\partial_\nu \psi(t_{n_k} + s^*), \varphi) + (\Delta \psi(t_{n_k} + s^*), \varphi) + (\Phi'(\psi(t_{n_k} + s^*)), \varphi) = 0.$$ 

(4.72)

Taking the limit $t_{n_k} \to \infty$ we obtain

$$a(\psi(t_{n_k} + s^*), \varphi) \to (\mu_\infty - \Phi'(\psi_\infty), \varphi)_{L_2}.$$ 

where $a : V \times V \to \mathbb{R}$ is the form defined in Section 2.5 and $(\cdot, \cdot)_{L_2}$ denotes the scalar product in $L_2(\Omega)$. Since $\Phi'(\psi_\infty) \in L_2(|\Omega)$ with $q = 6/5 + 2$ it follows that $\psi_\infty \in D(Aq) = \{ u \in H^2_0(\Omega) : \partial_\nu u = 0 \}$, where $Aq$ is the part of the operator $A$ in $L_q(\Omega)$ which is induced by the form $a(u, v)$. Observe that $q > 6/5$ by assumption, whence we may apply a bootstrap argument to conclude $\psi_\infty \in H^2_0(\Omega)$ and $\partial_\nu \psi_\infty = 0$ on $\Gamma$. Going back to (4.72) we obtain for $(t_{n_k})$ not $\infty$ the identity

$$\nabla \psi_\infty, \nabla \varphi_{L_2} + (\Phi'(\psi_\infty), \varphi) = (\mu_\infty, \varphi)_{L_2},$$

for all functions $\varphi \in V$. This yields (iii) after integration by parts. Assertion (iv) follows from (iii) and again via integration by parts, since by Proposition 2.5.2 the first Fréchet derivative of $E$ is given by

$$\langle E'(u), h \rangle_{V^*, V} = \int_\Omega \nabla u \nabla h \, dx + \int_\Omega \Phi'(u) h \, dx.$$
The next proposition is the key for the proof of the convergence of the orbit $\psi(t)$ towards a stationary state as $t \to \infty$.

**Proposition 4.7.3** (Lojasiewicz-Simon inequality). Let $\varphi \in V$ be a critical point of the functional $E$. Assume in addition that $\Phi$ is real analytic. Then there exist constants $s \in (0, \frac{1}{2}]$, $C, \delta > 0$ such that

$$|E(u) - E(\varphi)|^{1-s} \leq C|E'(u)|_V^s,$$

whenever $|u - \varphi|_V \leq \delta$.

**Proof.** The proof follows the lines of the proof of Proposition 2.5.4. We skip the details. \qed

Now we are in a position to state the main result of this section.

**Theorem 4.7.4.** Let $\Phi$ satisfy the conditions (4.67) and (4.68). Assume in addition that $\Phi$ is real analytic. Then the limit

$$\lim_{t \to \infty} \psi(t) =: \psi_\infty$$

exists in $V$ and $\psi_\infty$ is a strong solution of the stationary problem (4.71).

**Proof.** Since each element $\varphi \in \omega(\psi)$ is a critical point of $E$, Proposition 4.7.3 implies that the Lojasiewicz-Simon inequality is valid in some neighborhood of $\varphi \in \omega(\psi)$. By Proposition 4.7.2 (ii) the $\omega$-limit set is compact, hence there exists $N \in \mathbb{N}$ such that

$$\bigcup_{j=1}^N B_{\delta_j}(\varphi_j) \supset \omega(\psi),$$

where $B_{\delta_j}(\varphi_j) \subset V$ are open balls with center $\varphi_j \in \omega(\psi)$ and radius $\delta_j$. Additionally in each ball the Lojasiewicz-Simon inequality is valid. It follows from Proposition 4.7.2 (i) and (ii) that the energy functional $E$ is constant on $\omega(\psi)$, i.e. $E(\varphi) = E_\infty$, for all $\varphi \in \omega(\psi)$. Thus there exists an open set $U \supset \omega(\psi)$ and uniform constants $s \in (0, \frac{1}{2}]$ $C, \delta > 0$ with

$$|E(u) - E_\infty|^{1-s} \leq C|E'(u)|_V^s,$$

for all $u \in U$. A well-known result in the theory of dynamical systems states that the $\omega$-limit set is an attractor for the orbit $\{\psi(t)\}_{t \in \mathbb{R}_+}$. To be precise this means

$$\lim_{t \to \infty} \text{dist}(\psi(t), \omega(\psi)) = 0 \quad \text{in } V.$$

This implies that there exists some time $t^* \geq 0$ such that $\psi(t) \in U$ for all $t \geq t^*$ and thus the Lojasiewicz-Simon inequality holds for the solution $\psi(t)$, i.e.

$$|E(\psi(t)) - E_\infty|^{1-s} \leq C|E'(\psi(t))|_V^s, \quad t \geq t^*. \quad (4.73)$$

Define a function $H : \mathbb{R}_+ \to \mathbb{R}_+$ by $H(t) = (E(\psi(t)) - E_\infty)^s$. Then with (4.70) and (4.73) it holds that

$$-\frac{d}{dt} H(t) = (E(\psi(t)) - E_\infty)^{s-1} \left( -\frac{d}{dt} E(\psi(t)) \right) \geq \frac{s}{2} \left( \frac{\|\partial_t \psi(t)\|_V^2 + |\nabla \mu(t)|_V^2}{(E(\psi(t)) - E_\infty)^{1-s}} \right) \geq C \frac{\|\partial_t \psi(t)\|_V^2 + |\nabla \mu(t)|_V^2}{|E'(\psi(t))|_V^s}. \quad (4.74)$$

Following the lines of Section 2.5 the first Fréchet derivative of $E$ in $V$ reads

$$\langle E'(u), h \rangle_{V^*, V} = \int_{\Omega} \nabla u \nabla h \, dx + \int_{\Omega} \Phi'(u) h \, dx,$$
for all \((u, h) \in V \times V\). Setting \(u = \psi(t)\) and making use of (4.69)_2 we obtain with the help of Hölder’s inequality, Poincaré’s inequality and integration by parts

\[
\langle E'(\psi(t)), h \rangle_{V^*, V} = \int_\Omega (\mu(t) - \bar{\mu}(t))h \, dx - \int_\Omega c \cdot \nabla \mu(t)h \, dx - \beta \int_\Omega \partial_t \psi(t)h \, dx \\
\leq C(|\nabla \mu(t)|_2 + |\partial_t \psi(t)|_2)|h|_2,
\]

(4.75)
since \(\text{div } c(x) = 0, \ x \in \Omega \) and \((c(x)|\nu(x)) = 0, \ x \in \partial \Omega\). Taking the supremum in (4.75) over all functions \(h \in V\) with norm less than 1 it follows that

\[
|E'(\psi(t))|_{V^*} \leq C(|\nabla \mu(t)|_2 + |\partial_t \psi(t)|_2).
\]

We insert this estimate into (4.74) to obtain

\[
-\frac{d}{dt} H(t) \geq C_\varepsilon ([\nabla \mu(t)]_2 + |\partial_t \psi(t)|_2).
\]

Integrating this inequality from \(t^*\) to \(\infty\) it follows that \(|\partial_t \psi(t)|_2, |\nabla \mu(t)|_2 \in L_1(\mathbb{R}^+), \) since \(H(t) > 0\). This implies that the limit \(\lim_{t \to \infty} \psi(t) := \psi_\infty\) exists firstly in \(L_2(\Omega)\) but by relative compactness also in \(V\). Finally, by Proposition 4.7.2 (iii) the limit \(\psi_\infty\) is a solution of the stationary problem (4.71). The proof is complete.

\[\square\]

### 4.8 Appendix

For \((z_0, z_1) \in \mathbb{R} \times \mathbb{R}^n\) Hypothesis (H) reads

\[
\beta z_0^2 + (d|z_1)z_0 + (Bz_1|z_1) \geq \varepsilon (z_0^2 + |z_1|^2),
\]

where \(d := a + c\). Observe that the left side of this inequality can be rewritten as

\[
\left(\sqrt{\beta}z_0 + \frac{1}{2\sqrt{\beta}}(d|z_1)\right)^2 + \left(B - \frac{1}{4\beta}(d \otimes d)\right)z_1|z_1|.
\]

For a fixed \(z_1 \in \mathbb{R}^n\) we choose \(z_0 \in \mathbb{R}\) in such a way that the squared bracket is equal to 0. Thus we obtain the estimate

\[
(\beta Bz_1|z_1) - \frac{1}{4}(d \otimes d)z_1|z_1) \geq \varepsilon \beta |z_1|^2,
\]

valid for all \(z_1 \in \mathbb{R}^n\). By the definition of \(d\) it holds that

\[
d \otimes d = a \otimes a + c \otimes c + a \otimes c + c \otimes a,
\]

hence we obtain the identity

\[
\beta B - \frac{1}{2}(a \otimes c + c \otimes a) = \beta B - \frac{1}{4}(d \otimes d) + \frac{1}{4}(a \otimes a + c \otimes c - a \otimes c - c \otimes a)
\]

\[
= \beta B - \frac{1}{4}(d \otimes d) + \frac{1}{4}(a - c) \otimes (a - c).
\]

Since the matrix \((a - c) \otimes (a - c)\) is positive semi-definite we finally obtain the assertion.