2 Modeling expectation formation

In this chapter, different approaches to modeling inflation expectations are presented. First, theoretical concepts of adaptive expectations are demonstrated. Then, the rational expectations hypothesis is discussed in details. The merits and demerits of rational expectations as well as different versions and different tests of this hypothesis are also considered. Finally, the learning approach and its role in macroeconomics are explained. Approaches to learning including educive learning, adaptive learning, and rational learning are also illustrated.

2.1 Theoretical concepts

2.1.1 Adaptive expectations

One of the most familiar traditional models of expectation formation is adaptive expectations. This model can be stated using the following equation, where $P_t^e$ is this period’s expected inflation; $P_{t-1}^e$ is last period’s expected inflation; and $P_{t-1}$ last period’s actual inflation:

$$P_t^e = P_{t-1}^e + \lambda (P_{t-1} - P_{t-1}^e)$$

with $\lambda$ being a value between 0 and 1. According to this hypothesis, current expectations of inflation reflect past expectations and an “error-adjustment” term. The parameter value of $\lambda$ depends on what we think about the likely source of last period’s error. If it was a permanent shift in the process forming $P$, then we set $\lambda = 1$ so that $P_t^e = P_{t-1}$. This is static expectations: this year’s inflation is expected to be the same as last year’s. If last period’s error was just due to a random event, we set $\lambda = 0$, so there is no adjustment, and we should not change expectations at all ($P_t^e = P_{t-1}^e$). People will change expected inflation if there is a difference between what they were expecting it to be last period and what it actually was last period. In fact, expected inflation is revised by some fraction of most recent forecast errors. If the expected inflation was, say 5 percent, but the actual inflation 10 percent, people raise their expectations by some fraction $\lambda$ of the difference between 5 and 10. Using the Koyck transformation, the equation (1) can be transformed into
\[ P_t^e = (1 - \lambda)P_{t-1} + \lambda (1 - \lambda)P_{t-2} + \lambda^2 (1 - \lambda)P_{t-3} + \lambda^3 (1 - \lambda)P_{t-4} \ldots \] (2)

Now we can examine the relationship between \( P_t^e \) and \( P_t \). Suppose that \( P_t \) has been constant for a long time at \( P_0 \). Then, suppose that at time period \( T \), the inflation jumps up to \( P_1 \) and stays there indefinitely. At \( T \), all the terms on the right-hand side of equation (2) are equal to \( P_0 \), so the expected inflation for \( T \) is given by \( P_0 \), that is \( P_T^e = P_0 \):

\[ P_T^e = (1 - \lambda)P_0 + \lambda (1 - \lambda)P_0 + \lambda^2 (1 - \lambda)P_0 + \lambda^3 (1 - \lambda)P_0 \ldots = P_0 \]

Once \( T \) is over, expectations are formed by equation (2) with \( t \) set equal to \( T+1 \). Therefore, the first term on the right-hand side for period \( T+1 \) is \( P_1 \):

\[ P_{T+1}^e = (1 - \lambda)P_1 + \lambda (1 - \lambda)P_0 + \lambda^2 (1 - \lambda)P_0 + \lambda^3 (1 - \lambda)P_0 \ldots \]

Since \( P_1 > P_0 \), it is easy to verify that \( P_1 > P_{T+1}^e > P_T^e = P_0 \). There is some correction in \( T+1 \) for the error made at \( T \), but is not complete. At the start of following period, two of the right-hand terms of equation (2) include \( P_1 \). The remaining error is again partly corrected but the absolute value of correction is less. This process continues until the second term on the right-hand side of equation (1) diminishes to make the difference \( (P_t - P_t^e) \) arbitrarily small.

There are merits and demerits of the adaptive expectations hypothesis (AEH). On the one hand, the hypothesis has the advantages of being simple to operate as a “rule of thumb”. It is at the best appropriate in a stable environment where the price level moves up and down in a fairly random fashion, with the possibility of somewhat more permanent shifts in the background. On the other hand, however, it has two disadvantages: first, it is a backward-looking approach (no account of fully-announced future policies). Second, it has systematic errors based on the previous forecast with some correction for previous forecast errors. Individuals do not systematically learn from previous forecast errors, they do ignore information that would help them improve the accuracy of their forecasts. Thus, the AEH assumes suboptimal behavior on the part of economic agents. For example, consider the Phillips curve equation:

\[ P_t = P_{t-1} - (U_{t-1} - U^*) + \varepsilon \]

\( P_t = \) Actual inflation at time \( t \)
\[ U^* = \text{Natural of unemployment} \]

Assume that (for simplicity):
\[ U^* = U_{t-1} = U_{t-2} = U_{t-3} \ldots \]

then
\[ P_t = P_{t-1} + \varepsilon_t \]

with adaptive expectations:
\[ P^e_t = \lambda P_{t-1} + (1 - \lambda) P^e_{t-1} \]
\[ = \lambda P_{t-1} + (1 - \lambda)(\lambda P_{t-2} + (1 - \lambda) P^e_{t-2}) + \ldots \]

If \( \lambda = 0.5 \)
\[ P^e_t = 0.5 P_{t-1} + 0.25 P_{t-2} + 0.125 P_{t-3} + \ldots \]
\[ = 0.5 P_{t-1} + 0.25[P_{t-1} - \varepsilon_{t-1}] + 0.125[P_{t-1} - \varepsilon_{t-1} - \varepsilon_{t-2}] + \ldots \quad (3) \]

Equation (3) shows that the AEH ignore past forecast errors in forming expectations. Under adaptive expectations, if the economy suffers from constantly rising inflation rates, people would be assumed to sequentially underestimate inflation. This may be regarded unrealistic- surely rational people would sooner or later realize the trend and take it into account in forming their expectations. Moreover, models of adaptive expectations never reach an equilibrium; instead they only move toward it asymptotically.

### 2.1.2 Rational expectations

#### The rise of Rational Expectations

The rational expectations hypothesis responds to this criticism by assuming that individuals use all information available in forming expectations. During the late 1960s, rational expectations economics started changing the face of macroeconomics. Robert Lucas, Tomas Sargent, and Neil Wallace started to dominate the macroeconomic discussion. Notions such as the Lucas critique, the Lucas supply curve, and the Sargent-Wallace policy irrelevance proposition became integral parts of the macroeconomics discourse.
There are different reasons behind the rise of rational expectations (RE). Sent (1998) argues that the main factors are as follows:

1. **Expiration of the Phillips curve**: in the late 1960s to early 1970s, policy makers used a trade-off between inflation and unemployment to lower unemployment. However, they faced high inflation rates accompanied by high unemployment rates in the 1970s. In other words, the result of policy making was higher inflation with no benefits in terms of lower unemployment. Rational expectations economists were able to explain the expiration of the Phillips curve. They, using the rational expectation hypothesis, demonstrated that government actions caused an adverse shift of the Phillips curve.

2. **Policy irrelevance**: orthodox prescriptions of economic policy crumbled, since much of the effectiveness of these policies were based on the government’s ability to fool people. Rational expectations economists asserted that people can foil government policies by learning their mistakes. They justified the ineffectiveness of government intervention in the context of the failure of traditional Keynesian policies in the 1970s. Also, they recognized the limitations of their profession maintaining that the economy would basically be stable if it were not subjected to the shocks administered by the government.

3. **Using available techniques**: rational expectation economists used sophisticated mathematical techniques in order to predict. They learned and used the techniques of intertemporal optimization developed by mathematicians and control scientists. They also improved the tools of optimal prediction and filtering of stochastic processes. Some of these techniques such as classical linear prediction theory¹ was developed in 1940s to 1950s but did not immediately become part of economists’ toolkits. However, Peter Whittle made more accessible to economists this theory that was heavily used by rational expectation economists. This delay explains the lagged effect of Muth’s contributions. Thus rational expectation economists were able to calculate rational expectation equilibria using new techniques.

4. **Restoring symmetry**: the hypothesis of adaptive expectations had been used heavily up until the late 1960s. According to this hypothesis, individuals used forecasting errors in revising their expectations. Econometricians were presumed to be fully knowledgeable whereas the agents were assumed to make systematic

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1. The mathematical theory for interpreting distributed lags in terms of economic parameters and incorporating the rational expectations hypothesis in economic models.
forecasting errors period after period. Thus there was an asymmetry among economists or econometricians and the agents in that econometricians fit models that forecast better than agents. Rational expectations hypothesis (REH) removed this asymmetry making the econometricians part of the agents’ behavior. Therefore, rational expectation economists placed econometricians and agents on an equal footing by postulating that forecasts made by the agents within the model were no worse than those the econometricians who had the model.

5. Optimizing over information: according to REH, optimization over perceptions implied agents did the best they could and formed their views of future using available information, including their understanding of how the economy works. Rational expectation theorists extended expectation theory into the optimizing behaviors theory. If perceptions were not optimally chosen, unexploited utility or profit-generating possibilities would exist within the system. Hence, these economists insisted on the disappearance of all such unexploited possibilities.

6. Endogenizing expectations: Keynes (1936) doubted that expectations could be modeled accurately. So he considered expectations as given. Also, Keynes followers assumed that people made guesses about the future by looking exclusively backward. In fact, the hypothesis of adaptive expectations is backward-looking in that it allows the possibility of systematic forecasting errors for many periods in succession. This is a suboptimal use of available information and is not consistent with the idea of optimization. Even though people used adaptive expectations, no widely accepted economic theory was offered to explain the amount of the adjustment parameter. The mechanism of rational expectations’ formation is endogenously motivated and expectations or forecasts are correct on average if errors individuals remain satisfied with their mechanism. This hypothesis asserted that the resulting predictions might still be wrong, but the errors would be random. If errors follow a pattern, they contain information that could be used to make more accurate forecasts. Therefore errors were presumed to cancel out when all individual expectations are added together.

7. Making public predictions: some authors believed that the rise of rational expectations could fight the threat of indeterminacy of economic outcomes. This indeterminacy resulted from this fact that making both self-falsifying and self-fulfilling predictions about people was possible. Since outcomes depended partly on what people expected those outcomes to be if people’s behavior depended on their
perceptions, economic systems were thought to be self-referential. This led some economists to despair that economic models could produce so many outcomes that they were useless as instruments for generating predictions. Rational expectations, however, was a powerful hypothesis for restricting the range of possible outcomes since it focused only on outcomes and systems of beliefs that were consistent with one another. Under rational expectations, correct public predictions could be made because rational expectations predictions were presumed to be essentially the same as the predictions of the relevant economic theory. Also, the hypothesis consisted of expectational response of the agents and the influence of predictions on behavior of the agents.

8. **Countering bounded rationality:** rational expectations theory was born at the same time in the same situation as the concept of bounded rationality, namely, in the 1960s at the Graduate School of Industrial Administration (GSIA) at Carnegie Mellon University. Holt, Modigliani, Muth, and Simon were colleagues and worked on the Planning of Control of Industrial Operation Project, which consisted of developing and applying mathematical techniques to business decision making. Though Simon and Muth had both participated in the project, Simon saw the strong assumption underlying this project as an instance of satisfying, whereas Muth saw this special case as a paradigm for rational behavior under uncertainty. Some argue that Muth, in his announcement of rational expectations, explicitly labeled this theory as a reply to the doctrine of Simon’s bounded rationality.

9. **Restricting distributed lags:** in the late 1960s, rational expectation economists were confronted with theoretical models that analyzed individual behavior in a context without uncertainty and randomness. At the same time, since they treated their data probabilistically, they had to incorporate uncertainty and randomness in optimizing economic theory and using the outcome to understand, interpret, and restrict the distributed lags that abounded in the decision rules of dynamic macroeconomic models. They promised to tighten the link between theory and estimation.

10. **Incorporating vector autoregression:** the final causal background of rational expectations is related to the belief that it created a connection between vector autoregressions and economic theory. Some argue the REH was able to revive theory by showing that vector autoregressions was not necessarily atheoretical and could provide a statistical setting within which the restrictions implied by theoretical models could be imposed. In particular, rational expectation theorists exploited cross-
equation restrictions to connect the vector autoregressive parameters of decision rules with theoretical parameters of taste, technology, and other stochastic environments.

**Rational expectations and processes**

The rational expectation hypothesis (REH) assumes economic variables are generated by recurring processes (Attfield et al, 1991). Over time, economic agents learn the process determining a variable and they will use this knowledge and all information available (that is related to the variable) to form expectations of that variable. As a result, the agents’ subjective probability distribution coincides with the objective probability distribution of events¹. In other words, the expectations of agents will be the same as the conditional mathematical expectations based on the true probability model of the economy. For example, suppose the value of variable \( Y \) in period \( t \) is determined by its own lagged value and by lagged values of other variables \( X \) and \( Z \) in the following way:

\[
Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 X_{t-1} + \alpha_3 Z_{t-1} \tag{4}
\]

where \( \alpha_0, \alpha_1, \alpha_2 \) and \( \alpha_3 \) are constant coefficients. Consider a person who, at the end of period \( t-1 \), is trying to form an expectation about the value that \( Y \) is going to take in period \( t \). She knows that the process determining \( Y \) is given by equation (4): knowledge of this process is said to be part of her information set at the end of period \( t-1 \). She also knows the values of all lagged variables of \( X, Y, Z \), that also are part of her information set at the end of period \( t-1 \). If she is rational, her expectation of what \( Y \) is going to be in period \( t \), on the basis of her information set at the end of period \( t-1 \), will be formed as follows:

\[
E_{t-1}Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 X_{t-1} + \alpha_3 Z_{t-1} \tag{5}
\]

where \( E_{t-1} \) is the expectation of \( Y_t \) formed on the basis of the information available at the end of period \( t-1 \). The rational expectation of \( Y_t \) formed at period \( t-1 \) (denoted as

1. This is the strong version of the rational expectations hypothesis, due to Muth, (Pesaran, 1987).
$E[Y_t | I_{t-1}]$ is the mathematical expectation of $Y_t$ conditional on the available information at the end of period t-1 ($I_{t-1}$). If $Y$ does indeed continue to follow the process shown in equation (5) then this person’s expectation will be perfectly accurate, the person’s forecasting or expectational error is zero. This result is not general because in this case we assumed the process determining $Y$ is deterministic. However, most processes in real world are stochastic; that is, they include an unpredictable element of randomness in human responses. One way to incorporate this element in equation (4) is to add to it a random term ($v_t$):

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 X_{t-1} + \alpha_3 Z_{t-1} + v_t \quad (6)$$

$v_t$ may be positive or negative. Since this variable is seen as the result of a large number of random factors affecting human behavior, it is natural to think of small values of $v_t$ rather than large values. In fact, we assume that variable $v_t$ has a probability distribution centered at zero and a constant, finite variance $\{\sigma_v^2\}$. The value of $v$ in period t is unknown at the end of period t-1; it is not part of the information set at period t-1. But it is clear that a rational forecaster has to form some expectation of the value that $v$ is going to take in period t. The rational expectation of $Y$ in accordance with equation (6) is as follows:

$$E_{t-1}Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 X_{t-1} + \alpha_3 Z_{t-1} + E_{t-1}v_t \quad (7)$$

where $E_{t-1}v_t$ is the expectation of $v_t$ formed on the basis of all the information available at the end of period t-1. The best guess a rational agent can make of $v_t$ is that it will equal its mean value $E_{t-1}v_t = 0$. Thus, the rational expectation of $Y$ in period t, based on information available at the end of period t-1, can be written as:

$$E_{t-1}Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 X_{t-1} + \alpha_3 Z_{t-1} \quad (8)$$

Thus the rational expectation of the variable $Y$ in period t is its mathematical expectation given the available information. Rational expectations, as Muth (1961) explained, should be generated by the same stochastic process that generates the variable to be forecast.
In equation (8), if the process determining Y remains unchanged, it follows that the expectational error will be the random component $v$ of $Y$:

$$Y_t - E_{t-1} Y_t = v_t$$

(9)

The general characteristics of Rational Expectations

A number of important implications follow from the fact that, if the process determining $Y$ is understood, the error of rational expectation of $Y$ is the same as the random component of the process determining $Y$. They are as follows:

(a) The errors of rational expectations are on average zero

It is clear from equation (9) that once the process determining $Y$ is allowed to be stochastic the rational expectation of $Y$ will not always be perfectly accurate, for the random component $v$ is inherently unpredictable. The best a rational forecaster could do is expect the mean value of $v$ and that is defined to be zero. In fact, the error may be positive, negative or zero. But on average or over a large number of periods the negative errors will cancel out with the positive ones, leaving an average error of zero.

(b) The errors of rational expectations exhibit no pattern

If expectations are rationally formed, the forecasting error will equal the random element in the process being forecast. This random variable, and hence forecasting error, may be surprises or news in the system. If it exhibits no pattern, then the forecasting error does not exhibit any pattern either. But what happen if $v$ exhibits a pattern in the following way:

$$v_t = \beta_1 v_{t-1} + \varepsilon_t$$

(10)

The current value of $v$ is linked to the previous period’s value of $v$. $\varepsilon_t$ is a random error with zero mean which can not be predicted on the basis of any information available at the end of period $t-1$; $\beta_1$ is a coefficient, the value of which lies between -1 and +1. If $v$ is being determined according to equation (10) then rational people will form their expectation of current period’s value of $v$ in accordance with that process.
And since the value of v in the previous period, t-1, will be part of the available information at the end of period t-1, it follows that the forecast of v will diverge from the actual value of v by an unknown, unpredictable element \( e_t \). The error term \( (e_t) \) exhibits no pattern and has a mean value of zero. Thus even if v does exhibit a pattern, the rational forecast of Y would, on average, still be correct and the forecasting error would exhibit no pattern. As for the timing of a change in the method of forming expectations, the rational expectations hypothesis suggests that as long as there is no change in the process determining a variable, the method of forming expectations will not change. But if the actual process determining a variable is known to have changed, then the method by which expectations are formed will change in line with it.

(c) Rational expectations are the most accurate expectations

Rational expectations is the most efficient method of forecasting in that the variance of the forecasting errors will be lower under rational expectations than under any other method of forecasting or forming expectations. Because forecasts of a variable on the basis of rational expectations hypothesis will use all available information on the process determining that variable. In other words, as expectations are formed the unpredictable part of Y can not regularly be predicted. So any method of expectation formation will be inaccurate to a degree determined by the likely range of values that v can take. But it is possible to be even more inaccurate by forecasting without reference or with only partial reference to the process determining the variable.

General critique of the rational expectations hypothesis

Criticisms of the REH are as follows (Attfield et al, 1991):

(a) The plausibility of rationality

REH assumes people to use all the information about the process determining a variable when forming expectations. Is it really plausible? Can we really assume that all decision-makers are intelligent enough to use and fully understand all the available information? In reality people often ignore economic matters. This criticism is that a major assumption behind rational expectations is implausible.
The advocates of REH respond to this criticism in this way: first of all the idea that the typical individual is capable of making the best of opportunities open to him is a common one in economics. For example, in demand theory it is assumed that the typical person chooses to consume goods at a point given by the tangency of an indifference curve and a budget constraint. The mathematics behind this choice strategy is highly sophisticated for most people. Yet it is assumed that people act as if they understand it. If such assumption leads to a theory which makes accurate predictions, then the assumption of mathematical awareness is thereby shown to be a useful one. People forming expectations use firms- who specialize in or provide the service of making economic forecasts- or government bodies-who make forecast public.

Some economists also criticize the role of rationality in REH. Advocates of the hypothesis state that the role of rationality has been used in REH in that the process of acquiring information has been carried out up to the point where the marginal cost of acquiring more information equals the marginal benefit of making more accurate forecasts. But this point does not necessarily correspond to the point at which the forecasting error is equal to the purely random component of the determining process. It may be that knowledge about some determining variable could be obtained and extra accuracy thereby achieved, but only at a price which it is not worth paying. In that case the forecasting error will tend to be absolutely greater than the random element in the determining process. Advocates of REH accept this criticism but they assert that for most purposes it is not of great significance. The reason for this is that forecasting errors themselves are observed at no cost. For example, any error in your forecast about the level of prices is observed as a costless side-effect of shopping. In other words, it must be worthwhile to exploit this information fully until its marginal benefit is zero.

(b) The availability of information

REH assumes that the process Y is known and that the values of variables in that process are known at the end of period t-1. But what happens if we do not know the process determining the variable (Y) and if we are not able to acquire the necessary information? Advocates of the REH state that it is true that people cannot automatically know which variables are important in the process determining Y but it is also true that the REH doesn’t claim that they do. What the hypothesis argues is
that on average and after a period of time, economic agents will learn from past experience what the process is. They will combine this developed knowledge with current available information to form their expectations\(^1\). For example, if, at the end of period t-1 the rational agent does not know the true value of X in period t-1, and if the value of X in period t-1 determines the value Y in this period, the agent will have to form expectations of the value X in period t-1. Suppose the process determining Y is as follows:

\[ Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 X_{t-1} + \alpha_3 Z_{t-1} + \nu_t \]  

(11)

Suppose that the value of \( X_{t-1} \) is unknown at the end of period t-1. And let the process determining X in any period t as follows:

\[ X_t = \beta_0 + \beta_1 V_{t-1} + \beta_2 W_{t-1} + \varepsilon_t \]  

(12)

where V and W are other variables, the \( \beta \)'s are coefficients, and \( \varepsilon \) is a random error term with mean zero. The rational forecast of the unknown value of X in period t-1 will be as follows:

\[ E_{t-1}X_{t-1} = \beta_0 + \beta_1 V_{t-2} + \beta_2 W_{t-2} \]  

(13)

\( E_{t-1}X_{t-1} \) will be used in place of \( X_{t-1} \) in equation (11). Thus if \( X_{t-1} \) is unknown the rational expectations of Y in period t will be:

\[ E_{t-1}Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 (\beta_0 + \beta_1 V_{t-2} + \beta_2 W_{t-2}) + \alpha_3 Z_{t-1} \]  

(14)

The forecasting error will therefore be given by:

\[ Y_t - E_{t-1}Y_t = \nu_t + \alpha_2 \varepsilon_{t-1} \]  

(15)

Since \( \nu_t \) and \( \varepsilon_{t-1} \) are random errors with means of zero, neither of which can be even partly predicted on the basis of any information available at the end of period t-1. The rational forecast or expectation of Y in equation (14) is, in general, the most accurate forecast.

\(^1\)Friedman(1979), criticizing the REH, asserted that what is typically missing in rational expectations models is a clear outline of the way in which economic agents derive the knowledge which then they use to formulate expectations meeting requirement.
(c) Limits to the applicability of rational expectation

Many important economic events can be seen as unique, or at least exceptional or unusual due to the particular political circumstances of day. In what sense can the REH be said to apply to these exceptional cases? The advocates of rational expectations assert that the REH can best be applied to variables or events which can be seen as a part of recurring process. However, this class of events may be a larger one than is commonly thought. For example, governments desire to have a high level of economic activity at the time of general elections and may switch some policies. Such switches of policy could be seen as part of a fairly regular and reasonably predictable process. So an event which could be portrayed as unique from another viewpoint may well be part of an underlying recurring process.

(d) Testability of REH

Some economists have criticized that REH is not testable. Rational expectations theorists state that there are several layers to this criticism. First, if REH is taken rather loosely to imply that people make the best of their available information, then it may always be possible to define the available information so that the hypothesis becomes immune to falsification. This criticism is valid if tests of REH tended to employ the loose form of the hypothesis. But if they tend to employ strong versions of the hypothesis in which people’s knowledge of the process determining a variable is assumed to be the same as the best estimate that can be made of that process by econometric techniques then this criticism is hardly a strong one. Because this assumption leads to predictions which are both clear and different from the predictions derived from other theories about expectations.

An important criticism is that expectations about a variable are almost always only part of a model. Thus there are joint tests of the REH itself and the rest of the model. If the model fails the tests to which it is subjected one can always ‘rescue’ the REH by arguing that it is the rest of the model which is wrong. It is at times possible to distinguish between the restrictions imposed on the data by REH itself and the restrictions imposed by the rest of the model. However, the usefulness of the REH, in this way, can be tested informally and less satisfactory. If, time after time, this kind of models were rejected then we can reject the REH.
The final type of criticism of testability of REH is what is known as ‘observational equivalence’. For many rational expectations models which ‘fits the data’ there will always be a non-rational expectations model which fits the data equally well. The data themselves cannot discriminate between two theories, which are therefore said to be observationally equivalent. The implication of this is that, even if a rational expectations model ‘passes’ conventional empirical tests, this does not necessarily imply that one should accept the hypothesis. Whether you do or do not, depends on whether you find it more ‘plausible’ than the non-rational expectations model on some other unspecified grounds.

(e) Multi rational expectations equilibria

The models of Muth and Lucas assume that at any specific time, a market or the economy has only one equilibrium (which was determined ahead of time), so that people form their expectations around this unique equilibrium. If there is more than one possible equilibrium at any time then the more interesting implications of the theory of rational expectations do not apply. In fact, expectations would determine the nature of the equilibrium attained, reversing the line of causation posited by rational expectations theorists.

(f) Ability of agents in action

In many cases, working people and business executives are unable to act on their expectations of the future. For example, they may lack the bargaining power to raise nominal wages or prices. Alternatively, wages or prices may have been set in the past by contracts that cannot easily be modified. (In sum, the setting of wages and prices of goods and services is not as simple or as flexible as in financial markets.). This means that even if they have rational expectations, wages and prices are set as if people had adaptive expectations, slowly adjusting to economic conditions.

Different versions of RE

Many definitions of rational expectations (RE) have been proposed since Muth (1961) published his seminal article on this concept. In its stronger forms, RE operates as a coordination devise that permits the construction of a “representative agent” having “representative expectations.” Generally, two definitions for RE is used are applied research: the weak form and the strong form.
Weak-form RE

The weak version of RE is independent of the content of the agent’s information set. Suppose there are N agents \(i=1,...,N\) in an economy and \(E_{t,i} Y_{t+k}\) denote agent i’s subjective (personal) expectation formed at the end of period t of \(Y_{t+k}\) \((k \geq 1)\). Also let \(E\left[Y_{t+k} \mid I_{t,i}\right]\) denote the objectively true expectation for \(Y_{t+k}\) conditional on the information available to agent i at the end of period t \((I_{t,i})\). The agents are said to have weak-form rational expectations for variable \(Y_{t+k}\) if the following condition holds:

\[
For each i = 1,..., N, \ E_{t,i} Y_{t+k} = E\left[Y_{t+k} \mid I_{t,i}\right] + \epsilon_{t,i} \text{ where } \epsilon_{t,i} \text{ are serially and mutually independent finite-variance error terms that satisfy } E[\epsilon_{t,i} \mid I_{t,i}] = 0.
\]

Weak-form RE has some features. First, it is applicable only if there are “objectively true” conditional expectations. Weak-form RE assumes that agents make optimal use of all available information. Second, it is consistent with the idea of “economically rational expectations”, proposed by Feige and Pearce (1976), in which agent’s information sets are the result of cost-benefit calculations by the agents regarding how much information to obtain. Finally, many economists are willing to use this version, as a useful benchmark assumption consistent with the idea that agents are arbitrageurs who make optimal use of information.

Strong-form RE

Muth (1961) used a stronger version of RE in that he placed a restriction on the information sets of agents in theoretical economic models. This version guarantees the existence of “objectively true” conditional expectations but at the cost of transforming RE into an incredible concept in relation to the form of expectations that real economic agents could reasonably be supposed to have.

Agents in a theoretical model of a multi-agent economy will be said to have strong-form RE if they have weak-form RE and, in addition, their information sets at the end of period t contain the following information:
(a) The true structural equations and classification of variables for the model, including the actual decision rules used by each private and public (government) agent to generate actions and/or expectations;

(b) The true values for all deterministic exogenous variables of the model;

(c) The true probability distributions governing all exogenous stochastic terms;

(d) Realized values for all endogenous variables observed by the modeler through the end of period t.

Strong-form RE has some interesting features. First, it is assumed that agents are smart and as well informed about the economy. The issue that agents know a priori the actual decision rules used by each other agent is incredible. This version can therefore be interpreted as an idealized Nash equilibrium¹ benchmark for agents’ expectations that agents may (or may not) eventually arrive at through some process of reasoning and/or learning.

Second, in practice theorists modeling economic systems assume that they have an extraordinary amount of information about the true working of the economy. As a result, under strong-form RE, economic agents are presumed to have a great deal more information than would actually be available to any econometrician who attempted to test these models against data (Sargent, 1993).

Third, many economists are uncomfortable with the more common assumption in the strong-form RE. Nevertheless, this version becomes more acceptable if it is viewed as a possible ideal limiting point for the expectations of boundedly rational agents with limited information who engage in learning in successive time periods.

Finally, considering perfect foresight² RE is interesting. Agents in a theoretical model of a multi-agent economy will be said to have perfect foresight RE if the following two conditions hold:

1. If there is a set of strategies with the property that no player can benefit by changing her strategy while the other players keep their strategies unchanged, then that set of strategies and the corresponding payoffs constitute the Nash Equilibrium.

2. It must be noted that perfect-foresight RE differs from the perfect foresight assumption used in “Walrasian equilibrium models.” In the latter kind of models, perfect foresight is the assumption that households and firms correctly foresee the market-clearing levels and solve their optimization problems conditional on these levels.
(a) Agents have strong-form RE;
(b) There are no exogenous shock terms affecting the economy, so that all expectations are correct without error, e.g. \( E_{t+i} Y_{t+k} = Y_{t+k} \)

There are some implications of strong-form RE. First, if there is a change in the way a variable moves, then the way in which expectations of this variable are formed also changes. For example, a change in the government’s monetary policy rule leads to a change in the movements of the Fed Funds rate. Second, forecasts are not always exactly correct, but forecast errors are not predictable in advance and they average out to zero. Third, two reasons why expectations can fail to be rational in the strong-form sense: (a) agents fail to use all available relevant information and (b) agents fail to make optimal use of all available relevant information.

**An example of strong-form RE**

Suppose an economy is described by the Lucas Model (Caplan, 2000):

\[(IS) \quad y_t = -ar_t + u_t \quad (1)\]
\[(LM) \quad m_t - p_t = by_t - ci_t + v_t \quad (2)\]
\[(Fisher equation) \quad i_t = r_t + E_t p_{t+1} - p_t \quad (3)\]
\[(AS) \quad y_t = y^* + \alpha (p_t - E_t-1 p_t) \quad (4)\]
\[(Monetary Policy Rule) \quad m_{t+1} = m_t + \varphi_{t+1} \quad (5)\]
\[(Strong-Form RE) \quad E_t p_{t+1} = E_t [p_{t+1} | I_t] \quad (6)\]

Where \( y_t \) = output, \( p_t \) = price level, \( m_t \) = money supply, \( r_t \) = real interest rate, \( i_t \) = nominal rate, \( u_t, v_t \), and \( \varphi_t \) = random variables with mean 0, \( y^* \) = potential output, \( E_t p_{t+1} \) = the subjective forward-looking expectation of representative agent at time \( t \) regarding the price level in period \( t+1 \), \( E_t [p_{t+1} | I_t] \) = the objectively true conditional expectation, \( I_t \) = information set that is available to the representative agent at the end of period \( t \) whose contents are assumed to be consistent with strong-form RE.
All variables are logs of their level values. In the period $t$ predetermined variables are $m_t$ and $E_{t-1}p_t$ for $t > 1$. The exogenous variables are: $y^*$, $u_t$, $v_t$ and $\phi_t$ ; the positive exogenous constants $a$, $b$, $c$, and $\alpha$ ; an initial value $m_0 = m_0 + \phi_1$ for the period 1 money supply $m_1$, where $m_0$ is exogenously given, and initial value for $E_0p_1$.

Model equation (6) is incomplete as it stands, in that the “true conditional expectation” on the right hand side needs to be determined in a manner consistent with strong-form RE. That is, given this expectation, the subsequent way in which the price level for period $t+1$ is actually determined by the model equations must conform to this expectation in the sense that the objectively true $I_t$-conditioned expectation of the model-generated solution for the price level in period $t+1$ must coincide with the expectation assumed for this price level in model equation (6). To complete this model with strong-form RE, we must solve a fixed point problem of the form $f(x) = x$, where $x = E [p_{t+1} \mid I_t]$. To determine the needed expectational form, $E [p_{t+1} \mid I_t]$, the method of undetermined coefficients is used.

Conjecture a possible solution form for $p_t$ as a parameterized function of other variables, where the parameter values are unknown. Then, determine values for these unknown parameters that ensure strong-form RE. For simplicity assume that $y^* = 0$. Combining model equations (1) through (4) plus (6) leads to

1. It must be noted that there is a problem for the RE solution, it is not unique. In fact, multi rational expectations are likely to exist for models that include equations that are nonlinear in the endogenous variables. This spreads some doubts about the “rationality” of these RE solutions. For example, consider the following model of an economy:

\[ y_t = a + b E_{t-1}y_t + \varepsilon_t, \quad t \geq 1, \quad a > 0, \quad 0 < b < 1, \quad E [\varepsilon_t \mid I_{t-1}] = 0 \]  

If a representative agent forms his expectations for $y_t$ in period $t-1$ in accordance with strong-form RE, that is, 
\[ E_{t-1}y_t = E [y_t \mid I_{t-1}] \]  
In this case the $y_t$ generating process in (1) takes the form 
\[ y_t = a + b E [y_t \mid I_{t-1}] + \varepsilon_t, \quad t \geq 1 \]  
The right side of equation (3) can be expressed as a function $M(x)$ of $x$, where $x = E [y_t \mid I_{t-1}]$. Taking conditional expectations of both sides of (3), one can obtains a relation of the form 
\[ x = E [M(x) \mid I_{t-1}] \]  
\[ \equiv f_t(x), \quad t \geq 1 \]  
Suppose that the RE solution for output of a model economy in period $t$ satisfies a fixed point problem having form (4) and that two distinct solutions $x'$ and $x''$ exist- that is, $f_t (x') = x'$ and $f_t (x'') = x''$. Thus, if all agents in the economy at the end of period $t-1$ anticipate output level $x'$ for period $t$, the objectively true expected output level for the economy in period $t$ will be $x'$; and if instead, all agents in the economy at the end of period $t-1$ anticipate output level $x''$ for period $t$, the objectively true expected output level for the economy in period $t$ will be $x''$. 

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\[ p_t = (1/1+c) m_t + (c/1+c) \mathbb{E}[p_{t+1} \mid l_t] - \beta [p_t - \mathbb{E}[p_t \mid l_{t-1}]] + w_t \]  

where \( \beta = q[(b+c/a)/(1+c)]; w_t = (1/1+c)(c_t/a)u_t - v_t \)

Suppose it is conjectured that the solution for \( p_t \) takes the form
\[ p_t = q_1 m_t + q_2 w_t + q_3 \phi_t, \quad t \geq 1, \]  

Lead equation (9) one period and taking conditional expectation of both sides:
\[ \mathbb{E}[p_t+1 \mid l_t] = q_1 \mathbb{E}[m_{t+1} \mid l_t], \quad t \geq 0, \]  

Taking conditional expectation of both sides of equation (5) leads to \( \mathbb{E}[m_{t+1} \mid l_t] = m_t \), hence
\[ \mathbb{E}[p_t+1 \mid l_t] = q_1 m_t, \quad t \geq 0 \]  

Now lag equation (11) one period and lag equation (5) one period to substitute \( m_t - \phi_t \) in for \( m_{t-1} \), thus obtaining
\[ \mathbb{E}[p_t \mid l_{t-1}] = q_1 [m_t - \phi_t], \quad t \geq 1 \]  

Combining equations (9) and (12), one then has
\[ p_t - \mathbb{E}[p_t \mid l_{t-1}] = [q_1 + q_3] \phi_t + q_2 w_t, \quad t \geq 1 \]  

Using equations (11) and (13) to substitute out for the expectations in the price equation (7) and combining terms leads to
\[ p_t = [(1/1+c) + (c/1+c)q_1] m_t + [1-\beta q_2] w_t - \beta [q_1 + q_3] \phi_t, \quad t \geq 1 \]  

Now we have two distinct equations-equations (9) and (14) that state \( p_t \) as linear function of \( m_t \), \( w_t \), and \( \phi_t \). To make these equations consistent, set the three coefficients in (9) equal to the three coefficients in (14). It yields:
\[ q_1 = 1; \]  
\[ q_2 = (1/1+\beta); \]  
\[ q_3 = - (\beta/1+ \beta); \]
Thus it follows that one possible solution for $p_t$ consistent with strong-form RE is:

$$p_t = m_t + (1/1+\beta)w_t - (\beta / 1+ \beta) \phi_t$$  \hspace{1cm} (18)

Equation (18) shows that the price level is directly proportional to the money supply, a positive function of investment shocks, a negative function of money demand shocks, and a negative function of unexpected money supply increases. The corresponding strong-form RE for $p_t$, to be substituted in on the right hand side of model equation(6), is then found by taking the $I_t$-conditional expectation of each side of equation (18) bumped up one period, which yields

$$E[p_{t+1} \mid I_t] = E[m_{t+1} \mid I_t] = m_t, \ t \geq 1$$  \hspace{1cm} (19)

Combining model equation (4) (with $y^* = 0$) with (18) and (19), it follows that the solution for period $t$ output consistent with strong-form RE is given by

$$y_t = \alpha[ (1/1+\beta) \phi_t + (1 /1+\beta) w_t ]$$  \hspace{1cm} (20)

Output is an increasing function not of money, but of unexpected money shocks as well as of shocks $u_t$ and $v_t$ to the IS and LM curves.

Equation (20) has some conclusions for economic policymaking. From Lucas’ point of view, if the Central Bank decides to lower the unemployment rate by an expansionary monetary policy, then according to the REH the policy will be ineffective. People will see what the Central Bank is doing and raise their expectations of future inflation. This is in turn will counteract the expansionary effect of the increased money supply. All that the Central Bank can do is to raise the inflation rate, with at most temporary decreases in unemployment.

**Different tests of REH**
Following Sargent (1993), four different tests of Muthian rationality may be distinguished. Letting $x_{t-k}^e$ signify the expectation reported in the survey for a variable $X_t$ made at time $t-k$.

1. **Unbiasedness**: the survey expectation should be an unbiased predictor of the variable. That is, a regression of the form

$$ x_t = a + b x_{t-k}^e + \varepsilon_t $$

Should yield coefficient estimates $a=0$ and $b=1$. This is necessary condition. A sufficient condition is as follows

$$ x_t - x_{t-k}^e = E_t = \mu_t + \varepsilon_t $$

The hypothesis to test is $\mu = 0$

2. **Efficiency**: the survey expectation should use information about the past history of the variable in the same way that the variable actually evolves through time. That is, in the two regressions,

$$ x_{t-k}^e = a_1X_{t-1} + a_2X_{t-2} + \ldots + a_nX_{t-n} + \varepsilon_t $$

$$ X_t = b_1X_{t-1} + b_2X_{t-2} + \ldots + b_nX_{t-n} + u_t $$

It must be true that $a_i = b_i$ for all $i$. This test is called orthogonality test. Another possibility for examining the efficiency property is that the forecast error is tested for serial correlation.

3. **Forecast error unpredictability**: The forecast error, that is, the difference between the survey expectation and the actual realization of the variable, should be uncorrelated with any information available at the time the forecast is made.

4. **Consistency**: when forecasts are given for the same variable at different times in the future, the forecasts should be consistent with one another. For example, in the regressions,

$$ x_{t-2}^e = c_1 x_{t-2}^e + c_2X_{t-2} + \ldots + c_nX_{t-n} + \varepsilon_t $$
\[ \begin{align*}
    t-1 x_t^e &= a_1 x_{t-1} + a_2 x_{t-2} + \ldots + a_n x_{t-n} + u_t \\
\end{align*} \]

It must be true that \( c_i = a_i \) for all \( i \).

These tests are different ways of testing properties of conditional expectations in that whether the reported survey expectations are consistent with being conditional expectations. For example, consider the efficiency test and suppose that \( a_1 \neq b_1 \). Substracting the first equation from the second yields the expression

\[ x_t - x_{t-1} e = \text{forecast error} = (a_1 - b_1) x_{t-1} \]

Since, by hypothesis \( a_1 \neq b_1 \), this implies that the forecast error is correlated with \( x_{t-1} \), which violates the orthogonality of conditional expectations as long as \( x_{t-1} \) is contained in the information set. Although it would be desirable for any expectation mechanism to satisfy at least some of these four properties, conditional expectations must satisfy all of them.

2.1.3 Learning processes

Role of learning in macroeconomics

Learning in macroeconomics refers to models of expectation formation in which agents revise their forecasting rules over time as new data becomes available. Learning plays a key rule in macroeconomics. Rational expectations can be assessed for stability under different kinds of learning such as least squares learning. Learning can be useful when there is a structural change in economy. Suppose a new government appears. Agents need to learn about the new regime. Besides, learning can be used as a selection criterion when a model has more than one equilibrium solution. (Bullard, 1991) Let us illustrate this point using a model of hyperinflation. Assume a government prints money to finance a constant budget deficit, then

\[ P_t G_t = M_t - M_{t-1} \tag{1} \]

where \( P_t \) is the price level, \( G_t = \bar{G}_t \) is the constant real deficit, and \( M_t \) is the money stock. Suppose the demand function for money is as follows
where \( E_t p_{t+1} = E_t (\log(\frac{P_{t+1}}{P_t})) \) is the expected rate of inflation and real output has been assumed constant. Considering equilibrium in the money market and substituting (2) into (1) will result in

\[
\bar{G} = f(E_t p_{t+1}) - f(E_{t-1} p_t) e^{-pi}
\]  

(since \( \log(\frac{P_t}{P_{t-1}}) = p_t, \frac{P_t}{P_{t-1}} = e^{p_t} \))

It can be shown that equation (3) has two RE equilibria: the high inflation equilibrium and the low inflation equilibrium. If we assume rational expectations, the high inflation equilibrium is locally stable and the lower one is unstable. These rankings will be reversed if we assume adaptive expectations. If it is considered that stability is not the appropriate selection criteria in a rational expectations model then there is no mechanism to choose between the two equilibrium solutions. In such cases, learning provides a selection criterion.

Researchers have frequently faced the issue of multiplicity of RE equilibria in nonlinear models. Assume a nonlinear model \( y_t = F(y_{t-1}^e) \) has the S-shape shown below
Figure 1.1: Multiplicity of solutions in nonlinear models

The multiple steady states $y = F(y)$ occur at the intersection of the graph of $F(.)$ and 45-degree line. This possibility can appear in models with monopolistic competition, increasing returns to scale production or externalities. Other specifications of this model can present multiple perfect foresight equilibria taking the form of regular cycles in addition to a steady state or sunspot equilibria, taking the form of a finite state Markov process (Evans and Honkapohja, 2001). An interesting question may now be posed: which of the steady states are stable under learning.

Approaches to learning

Following Evans and Honkapohja (1999, 2001), the approaches to learning can be categorized into three groups: eductive learning, adaptive learning, and rational learning.

2.1.3.1 Eductive learning

In the eductive approach, we examine whether expectations converge to the rational expectations equilibrium through a process of reasoning. Consider the following example based on Decanio(1979)

Consider the demand and supply in a market are given by

$$q_i = a - bp_i + w_i$$

(4)
\[ q_t = c + dp_t^e + v_t \]  \hspace{1cm} (5)

Here \( q_t \) and \( p_t \) are the actual quantity and price level, \( w_t \) and \( v_t \) are random disturbances which are assumed to be white noise and \( a, b, c, \) and \( d \) are constant. Demand is downward-sloping linear function of the market price and supply depends positively and linearly on expected price due to a production lag. \( p_t^e \) denotes the expectations of the representative supplier (average expectations). The good is assumed to perishable and markets clear. The reduced form for the prices is given by

\[ p_t = A - Bp_t^e + u_t \]  \hspace{1cm} (6)

where \( A = \frac{a-c}{b}, B = \frac{d}{b}, \) and \( u_t = \frac{w_t - v_t}{b}. \)

First we examine the model under RE. The RE hypothesis can be formally stated as

\[ E(p_t|I_{t-1}) = E_{t-1}p_t \]  \hspace{1cm} (7)

So that expectations are the true mathematical conditional expectations, conditional on available information at the end of period \( t-1. \) The information set includes past data \{ \( u_{t-1}, u_{t-2}, \ldots, P_{t-1}, P_{t-2}, \ldots \} = I_{t-1} \) and knowledge of the model. We can compute RE by substituting (7) in (6) and obtain

\[ p_t = A - BE_{t-1}p_t + u_t \]  \hspace{1cm} (8)

Taking conditional expectations \( E_{t-1} \) of both sides yields \( E_{t-1}p_t = A - BE_{t-1}p_t \) so that expectations are given by

\[ E_{t-1}p_t = \frac{A}{1+B} \]

And the unique RE solution is of this form: \( p_t = \frac{A}{1+B} + v_t. \)

The RE equilibrium for the model is a random variable that is of the form constant plus noise. Under RE the appropriate way to form expectations depends on the stochastic process followed by the exogenous variables, \( v_t \) in this case.

Now we consider the model under eductive learning. Suppose agents form their expectations initially in an arbitrary manner, for example, static expectations

\[ E_{t-1}^0p_t = p_{t-1} \]  \hspace{1cm} (9)
The question is whether they can modify their behavior so that rational expectation equilibrium, given by $\frac{A}{1+B}$, would be attainable. Plugging (9) into (8) results in the actual evolution of prices

$$p_t = A - Bp_{t-1} + u_t$$

(10)

It is assumed that after some passage of time agents realize (reason or deduce) that prices are evolving according to (10) and form new expectation

$$E_{t-1}^1 p_t = A - Bp_{t-1}$$

(11)

The evolution of the system is changed by this new expectation

$$p_t = A - B(A - Bp_{t-1}) + u_t = A - BA + B^2 p_{t-1} + u_t$$

Observing the new evolution of prices, agents revise their expectations to

$$E_{t-1}^2 p_t = A - BA + B^2 p_{t-1}$$

(12)

So that prices evolve as follows by plugging (12) into (8)

$$p_t = A - B(A - BA + B^2 p_{t-1}) + u_t = A - BA + B^2 A - B^3 p_{t-1} + u_t$$

(13)

If we repeat this process, the expectations after n iterations will be

$$E_{t-1}^n p_t = A - BA + B^2 A - B^3 A + ... + B^n A + B^n p_{t-1}$$

(14)

$$= A(1 - B + B^2 - B^3 + ... + B^n) + B^n p_{t-1}$$

Since $(1 - B + B^2 - B^3 + ... + B^n = \frac{1}{1 + B})$ and $B^n p_{t-1} \to 0$ for $|B|<1$ and large $n$, expectations will converge to rational expectations

$$E_{t-1}^n p_t = \frac{A}{1+B}$$

The rational expectations, in this case, is said to be iteratively E-stable. It is clear that convergence to rational expectations is not guaranteed if $|B|>1$. Guesnerie, 1992; Evans, 1985, 1986; Peel and Chappell, 1986; and Bullard and Mitra, 2000), employing the iterative expectations method in different models, examined convergence to rational expectations.

2.1.3.2 Adaptive learning

Agents would learn from data via regression about the model and the policy regime. Although this would produce expectations formation very similar to adaptive
expectations, it will not necessarily ever converge to rational expectations (Benjamin Friedman, 1979). Bray and Savin (1986) and Fourgeaud, Gourieroux and Pradel (1986) initially applied least-squares learning mechanism to see whether it would converge to rational expectations. Here, for simplicity, it will be assumed that the reduced form for prices is as follows

\[ p_t = A - Bp_t^e + Cz_{t-1} + u_t \]  (15)

where \( z_{t-1} \) denotes observable exogenous variables. The rational expectations will be

\[ E_t^{\pi} = \frac{A + Cz_{t-1}}{1 + B} \]

and prices evolve as

\[ p_t = \frac{A}{1 + B} + \frac{C}{1 + B} z_{t-1} + u_t = \alpha + \beta z_{t-1} + u_t \]  (16)

where \( \alpha = \frac{A}{1 + B} \) and \( \beta = \frac{C}{1 + B} \).

It should be noted that this model has a unique RE since \( p_t \) does not depend on expected future prices. Now assume that agents know the true model but are unaware of the parameter values \( \alpha \) and \( \beta \). According to least-squares learning, agents are assumed to run least-squares regressions of \( p_t \) on \( z_{t-1} \) and an intercept. Rational forecast will be generated from the estimated model \( E_t^{\pi} = \alpha + \beta z_{t-1} \).

Agents revise the expectations by reestimating the model as more data becomes available. The coefficients \( (\alpha, \beta) \) will converge to the unique RE \( (\alpha, \beta) \) if \( B < 1 \). The conditions for convergence of recursive least-squares expectations \( (B < 1) \) can be weaker than those under iterative expectations \( (|B| < 1) \).

Agents perceive the reduced form as

\[ y_t = \beta x_t + e_t \]  (17)

Where the least squares estimated coefficients are given by

\[ \beta_t = \left( \sum_{i=0}^{t-1} x_i y_i \right) \left( \sum_{i=0}^{t-1} x_i^2 \right)^{-1} \]  (18)

The recursive least-squares estimates can be shown to be

\[ \beta_t = \beta_{t-1} + \gamma_t R_{t-1}^{-1} x_{t-1} (y_{t-1} - \beta_{t-1} x_{t-1}) \]  (19)

and

\[ R_t = R_{t-1} + \gamma_t (x_{t-1} y_{t-1} - R_{t-1}) \]  (20)
with the gain $\gamma_i = \frac{1}{t}$, an important factor in determining the speed of convergence to the true parameter, and where $R_i$ is an estimate of the moment matrix for $x_i$. For suitable initial conditions $R_i = t^{-1} \sum_{i=0}^{t-1} x_i x_i'$. 

Considering the recursive least-squares of the mean $Ez_i = \mu$ can help to understand the least-squares updating formula. The least-squares estimate is the sample mean $\bar{z}_i = \frac{1}{t} \sum_{n=1}^{t} z_n$. If we subtract the sample mean at $t-1$ from both sides of $\bar{z}_i$ and rearrange, then

$$\bar{z}_i = \bar{z}_{i-1} + \frac{1}{t} (z_i - \bar{z}_{i-1})$$

(21)

Since

$$\bar{z}_{i-1} = \frac{1}{t-1} \sum_{n=1}^{t-1} z_n$$

$$t \bar{z}_i = \frac{1}{t} \sum_{n=1}^{t} z_n = z_i + \sum_{n=1}^{t-1} z_n$$

$$(t-1) \bar{z}_{i-1} = \sum_{n=1}^{t-1} z_n$$

$$E_i - (t-1)z_{i-1} = z_i \rightarrow t(\bar{z}_i - \bar{z}_{i-1}) = z_i - \bar{z}_{i-1} \rightarrow \bar{z}_i - \bar{z}_{i-1} = \frac{1}{t} (z_i - \bar{z}_{i-1})$$

$$\bar{z}_i = \bar{z}_{i-1} + \frac{1}{t} (z_i - \bar{z}_{i-1})$$

Adaptive methods of learning have the same structure which is given by

$$\theta_i = \theta_{i-1} + \lambda_i Q(\theta_i, \theta_{i-1}, X_i)$$

(22)

Where $\lambda_i = \frac{1}{t}$ in the case of least-squares, $\theta$ is a vector of parameters, $Q$ is a function and $X_i$ is the vector of variables in the structural model. Adaptive expectations is in fact a special case of least-square adaptive learning (21) if the gain parameter $(\lambda_i = \lambda)$ is constant.

The evolution of $X_i$ will depend on $\theta_{i-1}$, in the case of a linear system

$$X_i = A(\theta_{i-1})X_{i-1} + B(\theta_{i-1})W_i$$

(23)

where $W_i$ is a vector of disturbance term.
Stability results for linear and nonlinear systems have been derived by Marcet and Sargent (1989a, 1989b), Evans and Honkapohja (1998). Ser gent (1999) asserts that if it is assumed that the US authorities employed constant-gain least-squares learning about the Phillips curve and maximized a social objective function to pick inflation, this fits US post-war data including the ‘great inflation’ well while rational expectations do not.

**Stability under adaptive learning**

When expectations are modeled by least-squares learning there is convergence to the rational expectation equilibrium (REE) as \( t \to \infty \) provided that that a stability condition is met. This condition can usually be obtained by the expectational stability (E-stability) approach. Consider the agents’ view of stochastic process for the market price as \( p_t = \alpha + \beta z_{t-1} + u_t \) which is called the *perceived law of motion* (PLM).

Expectations are based on the PLM and hence given by \( p_t^e = \alpha + \beta z_{t-1} \), where \( (\alpha, \beta) \) may not be the REE values. Agents are boundedly rational because they do not initially know parameters \( (\alpha, \beta) \) and they try to learn the REE solution over time. Inserting the PLM into the reduced form yields the corresponding *actual law of motion* (ALM):

\[
p_t = (A - B\alpha) + (C - B\beta)z_{t-1} + u_t
\]

This ALM has the same form as the PLM but with different values of the parameters. In fact, the above equation yields a mapping from the PLM parameters \( (\alpha, \beta) \) into the ALM parameters \( T(\alpha, \beta) = (A - B\alpha, C - B\beta) \).

Only at the REE values does one have \( T(\alpha, \beta) = (\alpha, \beta) \). Expectational stability looks at whether the REE is the stable outcome of a process that parameters of the PLM are adjusted slowly toward the parameters of the ALM that they induce. This adjustment is described by a differential equation and E-stability corresponds to local stability of the REE under these dynamics.

Consider a vector version of the model. Using the \( T \) mapping, E-stability is defined by the ordinary differential equation

\[
\frac{d}{d \tau} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = T \begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \begin{bmatrix} \alpha \\ \beta \end{bmatrix}
\]
An REE is said to be E-stable if it is a locally asymptotically stable fixed point (or equilibrium point or steady state) of this differential equation. Here $\tau$ denotes virtual time (it is distinct from real time $t$ and is measured in discrete periods). Plugging in the form of the mapping the system of differential equations will then be:

$$\frac{d\alpha}{d\tau} = A - (B + 1)\alpha, \quad (26)$$

$$\frac{d\beta_i}{d\tau} = C_i - (B + 1)\beta_i, \text{ for } i = 1, 2, \ldots, n, \quad (27)$$

where $n$ is the dimension of the vector of exogenous variables. Clearly, the unique fixed point is E-stable if $B<1$. Least-squares learning converges locally to an REE if and only if that REE is E-stable. Intuitively, a model is stable or learnable if the new data generated by one more observation under learning is on average closer to the REE than the current belief derived from past data.

### 2.1.3.3 Rational learning

The rational approach to learning recognizes the benefits and costs of more accurate forecasts for an agent so that rational expectations may not be achieved unless calculation costs are zero (Feige and Pierce, 1976; Evans and Ramsey, 1992). However the widely used method to model rational learning has been based on Bayes’ theorem. It is a method of updating belief, implying that beliefs change by learning. Data or new facts only influence the posterior belief, $P(A|B)$, through the likelihood function $P(B|A)$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad (28)$$

where $P(A)$ is prior belief. Many researchers used Bayes’ rule to model learning in the economic literature including learning about a new regime (see Cyert and Degroot, 1974; Backus and Driffill, 1985; Lewis, 1998; and Ellison and Valla, 2000). Consider the following example presented by Lewis (1988).

Assume the reduced form for the exchange rate is given by

$$s_t = m_t + \alpha(E_{t} s_{t+1} - s_t) \quad (29)$$

where $m_t$ is the money supply at time $t$, $s_t$ exchange rate and $\alpha$ is a positive constant. Also assume the money supply is as follows

34
\[ m_t = \theta_0 + \varepsilon_i^0 \]  

(30)

where \( \theta_0 \) is a constant and \( \varepsilon_i^0 \sim N(0, \sigma_i^2) \). Suppose at \( t=0 \) agents come to believe that the supply money process may have changed due to a new regime. The new process has the same form (30) except with different mean and variance:

\[ m_t = \theta_1 + \varepsilon_i^1 \chi_t \geq 0 \]  

(31)

We assume \( \theta_1 < \theta_0 \) and \( \theta_1 = 0 \), so that the process can be interpreted as going from ‘loose’ to ‘tight’ money. It is also assumed agents believe that if the policy has changed it will not be changed back and they also know the parameters of the potential new process. We can obtain the solution by solving (31) forward

\[ s_t = (1 - \gamma) \sum_{i=0}^{\alpha} \gamma^i E_i m_{t+i} \]  

(32)

where \( \gamma = \frac{\alpha}{1+\alpha} \)

Expected money supply equals

\[ E_i m_{t+i} = \theta_0 (1 - P_{1,t}) \] for any \( i>0 \) and \( t \geq 0 \)  

(33)

where \( P_{1,t} \) is agents’ assessed probability at time \( t \) that the process changed at time \( 0 \).

Finally the exchange rate is obtained as

\[ s_t = (1 - \gamma) m_t + \gamma (1 - P_{1,t}) \theta_0 \]  

(34)

To obtain the best estimate of \( P_{1,t} \), agents combine their prior beliefs about the probability together with their observations of money outcomes each period to update their posterior probabilities according to the Bayes’ rule

\[ P_{1,t} = \frac{P_{1,t-1} f(I_t | \theta_i)}{P_{1,t-1} f(I_t | \theta_i) + P_{0,t-1} f(I_t | \theta_0)} \]  

(35)

where \( P_{0,t} \) is the conditional probability of no change at \( t=0 \), \( f(I_t | \theta_i) \) is the probability of observing the information set \( I_t \) given that \( m_t \) follows the \( i \)th process. The ratio of posterior probabilities of each process, the posterior odds, is given by

\[ \frac{P_{1,t}}{P_{0,t}} = \frac{P_{1,t-1} f(m_t | \theta_i)}{P_{0,t-1} f(m_t | \theta_0)} = \left[ \frac{P_{1,t-1}}{P_{0,t-1}} \right] \left[ \frac{(1/\sigma_i^2) \exp(-1/2 \left[ m/\sigma_i^2 \right]^2)}{(1/\sigma_0^2) \exp(-1/2 \left[ m - \theta_0 \right]/\sigma_0^2)} \right] \]  

(36)
The first term on the right-hand side of equation (36) indicates that the change from t-1 to t in the relative conditional probabilities depends on the observation of the current money supply at time t. For instance, for some observation of current money supply, say \( m \), the probability of being under either policy process is the same; i.e., \( f(m|\theta_i) = f(m|\theta_0) \), so that the posterior probabilities, \( \frac{P_{1,t}}{P_{0,t}} \), equal the prior probabilities, \( \frac{P_{1,t-1}}{P_{0,t-1}} \), and therefore the conditional probabilities do not change.

However, observations of money different from \( m \) convey information, the last term on the right-hand side of (36), about the regimes causing probabilities to be revised. To analyze the behavior of the probabilities, equation (36) can be written as

\[
\log\left(\frac{P_{1,t}}{P_{0,t}}\right) = \log\left(\frac{P_{1,t-1}}{P_{0,t-1}}\right) + \log\left(\frac{f(m_t|\theta_i)}{f(m_t|\theta_0)}\right)
\]

Equation (37) is a linear difference equation in the dependent variable, \( \log\left(\frac{P_{1,t}}{P_{0,t}}\right) \). For simplicity assume that \( \sigma_1 = \sigma_0 = \sigma \), then

\[
\log\left(\frac{f(m_t|\theta_i)}{f(m_t|\theta_0)}\right) = \left[\frac{(m_k - \theta_0)^2 - m_k^2}{2\sigma^2}\right]
\]

(38)

Given the initial probabilities \( P_{1,0} \) and \( P_{0,0} \), and plugging (38) into (37), we obtain the solution to the difference equation as

\[
\log\left(\frac{P_{1,t}}{P_{0,t}}\right) = \log\left(\frac{P_{1,0}}{P_{0,0}}\right) + \sum_{k=1}^{t} \left[\frac{\theta_0^2 - 2m_k\theta_0}{2\sigma^2}\right]
\]

(39)

Equation (39) indicates that the behavior of the probabilities depends on the actual observations of the process. For example, when the money supply observed today is strongly negative, agents think it is more likely that policy has changed.

Taking expectations of (39) and defining \( \theta_i \) as the true \( \theta \) gives

\[
\mathbb{E}\log\left(\frac{P_{1,t}}{P_{0,t}}\right) = \log\left(\frac{P_{1,0}}{P_{0,0}}\right) + t \left[\frac{\theta_0^2 - 2\theta_0\theta_i}{2\sigma^2}\right]
\]

(40)

Equation (40) shows that the expected value of the ‘true’ process rises over time. For example, if policy has changed so that \( \theta_i = \theta = 0 \), then the log probability increases
to infinity due to the term \( \frac{t\theta_0^2}{2\sigma^2} \) as \( t \) goes to infinity. Similarly, when policy has not changed so that \( \theta_i = \theta_0 \) the log probability goes to negative infinity due to the term \( \frac{-t\theta_0^2}{2\sigma^2} \) as \( t \) goes to infinity. Also, it can be demonstrated that the expected value of the log ratio of probabilities converges and its speed depends positively on \( \frac{\theta_0^2}{\sigma^2} \).

Therefore, the speed of market learning depends upon the squared signal-to-noise ratio.

Using the above analysis, Lewis (1988) investigates the effects of the probability behavior on the exchange rate and forecast errors. Taking expectation of the exchange rate at \( t-1 \) and subtracting from (34) we obtain the forecast errors of exchange rate corresponding to each potential process

\[
s_i - E_{t-1}s_i = (1 - \gamma)c_i^0 + \theta_0(P_{t-1} - \gamma P_{t-1}) \quad \text{if } \theta_i = \theta_0
\]

\[
s_i - E_{t-1}s_i = (1 - \gamma)c_i^1 + \theta_0(P_{t-1} - \gamma P_{t-1}) \quad \text{if } \theta_i = \theta_1
\]

The expected value of the last component of the equations (41) and (42) shows dependence on the conditional probabilities. Whilst agents are learning, the evolution of these probabilities depends on the random observations of the money process, and does not equal the true values.

Taking expectations of forecast errors in equation (42) conditional upon a change in policy to \( \theta_1 \) and initial probabilities, gives the expected evolution of forecast errors, for a large number of \( m_k \) as

\[
E(s_i - E_{t-1}s_i | \theta_1) = -\theta_0[E(P_{t-1} | \theta_1) - \gamma E(P_{t-1} | \theta_1)] < 0
\]

The inequality is negative since the discount rate, \( \gamma \), is less than one and \( E(P_{t-1} | \theta_1) < E(P_{t-1} | \theta_1) \). Hence, if agents do not completely realize that the policy has changed to a ‘tighter’ money supply process, the exchange rate will be expected to be weaker than subsequently occurs. Lewis’ model shows well how learning about a regime change using Bayes’ rule can imitate the outcomes of the Peso problem¹.

1. The peso problem, which was initially examined by Milton Friedman in his analysis of the behavior of the Mexican currency, refers to a situation where rational agents anticipate the possibility of future changes in the data-generating mechanism of economic variables.