A basic ingredient for results in the spirit of Ekeland’s variational principle is a metric space and an order relation on the space defined in terms of the metric itself. If a function is involved that maps not into the reals but into a more general set, for example a linear space, the metric has to be replaced by an expression mapping into the same set. On the other hand, not all properties of a metric are really essential for a proof of a variational principle. This has been already realized in [8]. Therefore, we extend the concept of a (real-valued) metric to functions into ordered monoids maintaining only a few but not all properties of a metric. We do not focus on topological structures which may be generated by such extensions of a metric as it has been done in Section 2.3 with the concept of order (pseudo)metrics. Of course, an order (pseudo)metric is an example of an order premetric that is introduced in the next definition.

**Definition 30** Let $X$ be a nonempty set and $(Y, \circ, \leq)$ a quasiordered monoid with neutral element $\theta \in Y$. A function $\Phi : X \times X \to Y$ is called an order premetric iff the following conditions are satisfied:

- (P1) $\forall x \in X: \theta = \Phi (x, x)$;
- (P2) $\forall x_1, x_2 \in X: \theta \leq \Phi (x_1, x_2)$;
- (P3) $\forall x_1, x_2, x_3 \in X: \Phi (x_1, x_3) \leq \Phi (x_1, x_2) \circ \Phi (x_2, x_3)$.

The condition (P1) is not a true restriction as the following lemma shows.

**Lemma 4** Let $X, Y$ be as in Definition 30 and $\Psi : X \times X \to Y$ be a function satisfying (P2) and (P3). Then the function $\Phi : X \times X \to Y$ defined by

$$\Phi (x_1, x_2) := \begin{cases} 
\Psi (x_1, x_2) & : x_1 \neq x_2 \\
\theta & : x_1 = x_2
\end{cases}$$

is an order premetric.
Proof. For \( \Phi \), the conditions (P1), (P2), (P3) may be checked straightforward. ■

Definition 31 Let \((X, \mathcal{U})\) be a uniform space and \((Y, \circ, \leq)\) a quasiordered monoid with neutral element \( \theta \in Y \). A function \( \Phi : X \times X \to Y \) satisfying (P2), (P3) of Definition 30 is called (sequentially) regular with respect to \( y_1, y_2 \in Y \) iff it satisfies:

(P4) If \( \{ x_n \}_{n \in \mathbb{N}} \subseteq X \) and

\[
\forall n \in \mathbb{N} : y_1 \circ \sum_{k=0}^{n} \Phi(x_{k+1}, x_k) \leq y_2,
\]

then \( \{ x_n \}_{n \in \mathbb{N}} \) is asymptotic, i.e.

\[
\forall E \in \mathcal{U} \exists n_E \in \mathbb{N} \forall n \geq n_E : (x_{n+1}, x_n) \in E.
\]

Note that, if \((X, d)\) is a metric space, a sequence \( \{ x_n \}_{n \in \mathbb{N}} \) is asymptotic if and only if \( \lim_{n \to \infty} d(x_{n+1}, x_n) = 0 \). The definition above applies also to this case.

Let \( Y \) be not only a monoid but also a group. Then \( y_1 \circ \sum_{k=0}^{n} \Phi(x_{k+1}, x_k) \leq y_2 \) for all \( n \in \mathbb{N} \) if and only if \( \theta \leq \sum_{k=0}^{n} \Phi(x_{k+1}, x_k) \leq y_2 \circ y_1^{-1} \) for all \( n \in \mathbb{N} \). Hence, it is enough to assume that the boundedness of above of \( \{ \sum_{k=0}^{n} \Phi(x_{k+1}, x_k) : n \in \mathbb{N} \} \) implies that \( \{ x_n \}_{n \in \mathbb{N}} \) is asymptotic.

Lemma 5 Let \((X, \mathcal{U})\) be a uniform space, \((Y, \circ, \leq)\) a quasiordered monoid and \( \Psi : X \times X \to Y \) be a function satisfying (P2) and (P3). Then, the order premetric \( \Phi \), defined via (3.1) is regular if and only if \( \Psi \) is regular.

Proof. Clearly, the regularity of \( \Phi \) implies the regularity of \( \Psi \). To show the converse, assume the regularity of \( \Psi \) and take a sequence \( \{ x_n \}_{n \in \mathbb{N}} \subseteq X \) such that

\[
\forall n \in \mathbb{N} : y_1 \circ \sum_{k=0}^{n} \Phi(x_{k+1}, x_k) \leq y_2.
\]

If \( x_{n+1} = x_n \) for some \( n \in \mathbb{N} \), we may delete \( x_{n+1} \) from the sequence since \( (x_{n+1}, x_n) \in E \) for each \( E \in \mathcal{U} \) in this case. Doing this as long as possible, we either obtain only finitely many elements of the original sequence or a subsequence \( \{ x_{n_l} \}_{l \in \mathbb{N}} \subseteq X \) such that \( x_{n_{l+1}} \neq x_{n_l} \). In the first case, the original sequence is constant up to finitely many elements and hence asymptotic. In the second case, we have \( \Phi(x_{n_{l+1}}, x_{n_l}) = \Psi(x_{n_{l+1}}, x_{n_l}) \) for all \( l \in \mathbb{N} \). This implies

\[
\forall l \in \mathbb{N} : y_1 \circ \sum_{k=0}^{l} \Psi(x_{n_{k+1}}, x_{n_k}) \leq y_2.
\]

From the regularity of \( \Psi \) we may deduce that for \( E \in \mathcal{U} \) and \( n_l \in \mathbb{N} \) sufficiently large, we have \( (x_{n_{l+1}}, x_{n_l}) \in E \). This completes the proof. ■
Example 13 In [65], Kada et al. introduced the concept of a \( w \)-distance as follows: Let \((X,d)\) be a metric space and \(w : X × X → \mathbb{R}_+\) be a function satisfying (i) \(w(x_1,x_3) ≤ w(x_1,x_2) + w(x_2,x_3)\) for all \(x_1,x_2,x_3 ∈ X\); (ii) For each \(x_0 ∈ X\), the function \(x → w(x_0,x)\) is lower semicontinuous; (iii) For each \(ε > 0\), there is \(δ > 0\) such that \(w(x,x_1) < δ, w(x,x_2) < δ\) imply \(d(x_1,x_2) < ε\).

We show that a \( w \)-distance is a regular premetric with \((Y,\circ) = (\mathbb{R}_+,+)\) and the usual \(≤\)-relation for real numbers for \(y_1 = 0\) and each \(y_2 = r ∈ \mathbb{R}_+\). To this purpose, take a sequence \(\{x_n\}_{n ∈ \mathbb{N}} ⊂ X\) such that \(\sum_{k=0}^n w(x_{k+1},x_k) ≤ r\) is true for all \(n ∈ \mathbb{N}\). Since \(0 ≤ w(x_{n+1},x_n)\) for all \(n ∈ \mathbb{N}\), this implies \(0 = \lim_{n→∞} w(x_{n+1},x_n)\). Fix \(ε > 0\). Then there is \(n_ε ∈ \mathbb{N}\) such that

\[
∀n ≥ n_ε : w(x_n,x_{n+1}) < \frac{δ}{2} < δ, \quad w(x_{n+1},x_{n+2}) < \frac{δ}{2}
\]

with \(δ > 0\) from (iii). By (i), we obtain

\[
w(x_n,x_{n+2}) ≤ w(x_n,x_{n+1}) + w(x_{n+1},x_{n+2}) < δ.
\]

and therefore from (iii) \(d(x_{n+1},x_{n+2}) < ε\) for all \(n ≥ n_ε\). Hence the sequence \(\{x_n\}_{n ∈ \mathbb{N}}\) is asymptotic.

In [65], a list of examples can be found showing that the set of \( w \)-distances contains the metric \(d\) but much more elements. Note that already in Brønstedts paper [8] similar functions has been used on uniform spaces.

Example 14 A simple example of an ordered monoid is \((Y := \mathbb{R}_+ ∪ \{+∞\},+,≤)\). Let \((X,d)\) be a metric space. Then \(d\) is a regular order premetric with respect to \(y_1 = 0, y_2 = r ∈ \mathbb{R}_+ ⊂ Y\), but not with respect to \(y_2 = +∞ ∈ Y\), of course.

Example 15 Let \((X,d)\) be a metric space and let \((Y,+,T,≤_K)\) be a normally ordered separated locally convex space with ordering cone \(K ⊆ Y\). Take \(k ∈ K \setminus \{0\}\). Then \(Φ(x_1,x_2) := kd(x_1,x_2)\) is a regular order premetric in the sense of Definition 30. This result is presented by Isac, compare Proposition 1 and the proof of Theorem 3 of [59].

Example 16 Let \((X,\mathcal{U})\) be a uniform space and let \((Y,\circ,≤,T)\) be a normally ordered topological Abelian group. Then, every order pseudometric \(D : X × X → Y\) in the sense of Definition 25 is an order premetric.