Chapter 6

Variational Principles on Complete Uniform Spaces

This chapter is devoted to variational principles on complete uniform spaces. The main difference to the metric case is the appearance of a transfinite induction argument such as Zorn’s lemma. Without additional assumptions, i.e., simply transforming Theorem 16 into the context of uniform spaces we are not able to avoid such an argument. The situation completely changes if a scalarization functional is present or can be constructed. This is the theme of the next chapter.

Minimal element theorem on uniform spaces are a common generalization of Phelps’ lemma (Lemma 1 in [101] from 1963) on the one hand and Ekeland’s variational principle from 1972 ([28], [30]) on the other hand. The former is in topological linear spaces, the latter in metric spaces, both classes of spaces belong to the class of uniform spaces. The first result in this direction is Theorem 1 in [8] due to Brønstedt. A very general result has been given by Vályi in the 1985 paper [124]. Besides, he proved also the first so called vector valued version of Ekeland’s principle on uniform spaces (Theorem 5 of [124]).

6.1 The basic theorem on complete uniform spaces

6.1.1 Preliminaries

Let \((X, \mathcal{U})\) be a uniform space with uniformity \(\mathcal{U} \subseteq \mathcal{P}(X \times X)\).

Let \(\preceq\) be a quasiorder on \(X\), i.e., a reflexive and transitive relation. As before, we denote the lower sections \(S_l(x) = \{x' \in X : x' \preceq x\}\) by \(S(x)\) for \(x \in X\), compare Definition 12.

Let \((A, \succ)\) be a directed set (compare [72], p. 65). A net \(\{x_\alpha\}_{\alpha \in A} \subseteq X\) is said to be decreasing with respect to \(\preceq\) iff

\[
\forall \alpha, \beta \in A, \alpha \succ \beta : x_\alpha \preceq x_\beta.
\]

In this chapter, \((X, \mathcal{U})\) is assumed to be complete. We note that the results can be modified in order to replace the completeness by \(\preceq\)-completeness as in Chapter 4.
A quasiorder $\preceq$ is called **regular** iff every decreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \subset X \) is asymptotic, i.e.,
\[
\forall E \in \mathcal{U}, \exists n_E \in \mathbb{N}, \forall n \geq n_E : (x_{n+1}, x_n) \in E.
\]
As in the case of a metric space, regularity forces antisymmetry.

**Proposition 42** A regular quasiorder $\preceq$ on a separated uniform space $X$ is antisymmetric.

**Proof.** Take $x, x' \in X$ such that $x \preceq x' \preceq x$. Then, the sequence \( \{x, x', x, x', \ldots\} \) is decreasing. Regularity implies
\[
\forall D \in \mathcal{U}: (x, x'), (x', x) \in E.
\]
Hence $x = x'$ since $X$ is separated.

A quasiorder $\preceq$ is called **lower closed** iff for any decreasing net \( \{x_\alpha\}_{\alpha \in A} \subset X \) converging to some $x \in X$
\[
\forall \alpha \in A: x \preceq x_\alpha
\]
holds true. A quasiorder is lower closed if and only if the sections $S(x)$ are closed with respect to decreasing nets, i.e. if \( \{x_\alpha\}_{\alpha \in A} \subset S(x) \) and \( \lim_\alpha x_\alpha = x \), then $x \in S(x)$.

**6.1.2 The basic theorem**

The following theorem is parallel to Theorem 16. A theorem of this type has been established by Vályi in [124].

**Theorem 24** Let the following assumptions be satisfied:

(M1) $(X, \mathcal{U})$ is a complete uniform space;
(M2) $\preceq$ is a reflexive and transitive relation on $X$;
(M3) $\preceq$ is regular;
(M4) $\preceq$ is lower closed.

Then, for each $x_0 \in X$ there exists $\bar{x} \in X$ such that
\[
\bar{x} \in S(x_0) \quad \text{and} \quad \{\bar{x}\} = S(\bar{x}).
\]

**Proof.** Consider the set $S(x_0) := \{x \in X : x \preceq x_0\}$. Let $S_0 \subset S(x_0)$ be a totally ordered subset of $S(x_0)$. Consider $S_0$ to be a decreasing net,
\[
S_0 = \{x_\alpha\}_{\alpha \in A}, \quad x_\alpha \preceq x_{\alpha'} \quad \text{for} \quad \alpha \succ \alpha'
\]
for some index set $A$, directed by $\succ$. We claim that \( \{x_\alpha\}_{\alpha \in A} \) is Cauchy. Assume the contrary. Then there exist $E \in \mathcal{U}$ and \( \{x_n\}_{n \in \mathbb{N}} \subset \{x_\alpha\}_{\alpha \in A} \subset S_0$ such that
\[
(x_{n+1}, x_n) \notin E \quad \text{for} \quad n \in \mathbb{N}.
\]
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Indeed, if \( \{x_\alpha\}_{\alpha \in A} \) is not Cauchy, there is \( E \in \mathcal{U} \) such that
\[
\forall \alpha \in A \exists \alpha_2 > \alpha_1 > \alpha : (x_{\alpha_1}, x_{\alpha_2}) \notin E.
\]
Hence we can find \( \alpha_1, \alpha_2 \in A \) such that \( \alpha_2 > \alpha_1 \) and \((x_{\alpha_1}, x_{\alpha_2}) \notin E\). Set \( x_1 := x_{\alpha_1} \), \( x_2 := x_{\alpha_2} \). Similarly, \( \alpha_3, \alpha_4 \in A \) can be found such that \( \alpha_4 > \alpha_3 > \alpha_2 \) and \((x_{\alpha_3}, x_{\alpha_4}) \notin E\). Set \( x_3 := x_{\alpha_3}, x_4 := x_{\alpha_4} \) and continue this procedure. A decreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) is obtained being not asymptotic. This contradicts (M3).

Since \( X \) is complete, \( \{x_\alpha\}_{\alpha \in A} \) converges to some \( \bar{x}_0 \in X \). From (M3) we obtain that \( \bar{x}_0 \in S(x_\alpha) \) for each \( \alpha \in A \), especially \( \bar{x}_0 \in S(x_0) \). Hence \( \bar{x}_0 \preceq x_\alpha \) for each \( \alpha \in A \), i.e., \( \bar{x}_0 \) is a lower bound of \( S_0 \).

By Zorn’s lemma, there exists a minimal element \( \bar{x} \) in \( S(x_0) \). Moreover, \( \{\bar{x}\} = S(\bar{x}) \) because if \( x \neq \bar{x}, x \preceq \bar{x} \) we obtain by transitivity \( x \preceq \bar{x} \preceq x_0 \) contradicting the minimality of \( \bar{x} \) in \( S(x_0) \).

\[\Box\]

**Remark 26** The uniform structure \( \mathcal{U} \) on \( X \) can be equivalently generated by a family of pseudometrics \( \{p_\lambda\}_{\lambda \in \Lambda} \) according to Definition 23. This means, each \( E \in \mathcal{U} \) contains a set of the form
\[
E_{\lambda,r} := \{(x,x') : d_\lambda(x,x') < r\}, \lambda \in \Lambda, r > 0.
\]
The sets \( E_{\lambda,r} \), \( \lambda \in \Lambda, r > 0 \) form a base of the uniform structure \( \mathcal{U} \) on \( X \). Hence, a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset X \) is asymptotic if and only if
\[
\forall r > 0, \forall \lambda \in \Lambda, \exists n_{r,\lambda} \in \mathbb{N}, \forall n \geq n_{r,\lambda} : p_\lambda(x_{n+1},x_n) < r.
\]
Similarly, the property of being asymptotic can be described by quasimetrics (see Definition 23) or an order metric (see Definition 25).

6.1.3 Equivalent formulations of the basic theorem

Without serious difficulties it is possible to transform the equivalent formulations of the basic minimal element theorem for metric spaces to the case of uniform spaces. We shall give the statements and refer for the details of the proofs to Section 4.1.3.

**Theorem 25** Let (M1) through (M4) of Theorem 24 be in force and, additionally, \( T : X \to \mathcal{P}(X) \) be a set-valued mapping. If \( T \) satisfies
\[
\forall x \in X, \exists x' \in T(x) : x' \preceq x, \tag{WC}
\]
then there is \( \bar{x} \in X \) such that \( \bar{x} \in T(\bar{x}) \), i.e. \( \bar{x} \) is a fixed point of \( T \). If \( T \) satisfies
\[
\forall x \in X, \forall x' \in T(x) : x' \preceq x, \tag{SC}
\]
then there is \( \bar{x} \in X \) such that \( \{\bar{x}\} = T(\bar{x}) \), i.e. \( \bar{x} \) is an invariant point of \( T \).

**Proof.** Each point \( \bar{x} \) satisfying the conclusions of Theorem 24 does the job. \[\Box\]
Theorem 26 Let (M1) through (M4) of Theorem 24 be in force and, additionally:

(M5) The set $M \subseteq X$ satisfies

$$\forall x \in S(x_0) \setminus M \; \exists x' \in S(x) \setminus \{x\}.$$ 

Then, there exists $\bar{x} \in S(x_0) \cap M$.

Proof. By Theorem 24, there exists $\bar{x} \in S(x_0)$ such that $\{\bar{x}\} = S(\bar{x})$. By assumption (M5), $\bar{x} \in M$, hence $\bar{x} \in M \cap S(x_0)$.

Theorem 24 can be derived from Theorem 25 and Theorem 26 in the same way as Theorem 16 from Theorem 17 and Theorem 18, respectively.

6.1.4 Set relation formulation

In this section, the analogues to the Theorems 21 and 22 shall be established.

Let $(X, \mathcal{U})$ be a uniform space and $Y$ as well as $M \subseteq X \times Y$ be nonempty sets. For $x \in X$, let us define $M(x) := \{(x', y) \in X \times Y : x' = x, (x', y) \in M\} \in \hat{P}(X \times Y)$ and $M_Y(x) := \{y \in Y : (x, y) \in M\} \in \hat{P}(Y)$. Let $\preceq$ be a quasiorder on $M$. Then, $\{(M_X : x \in X), \preceq\}$ as well as $\{(M(x) : x \in X), \preceq\}$ is quasiordered. As in Section 4.1.6 we have $M(x') \preceq M(x)$ if and only if

$$\forall y \in M_Y(x), \exists y' \in M_Y(x') : (x', y') \preceq (x, y)$$

(6.1)

and $M(x') \preceq M(x)$ if and only if

$$\forall y' \in M_Y(x'), \exists y \in M_Y(x) : (x', y') \preceq (x, y).$$

(6.2)

Theorem 27 Let the following assumptions be satisfied:

(M1') $(X, \mathcal{U})$ is a uniform space and $X, Y$ as well as $M \subseteq X \times Y$ are nonempty sets;

(M2') $\preceq$ is a quasiorder, i.e. a reflexive and transitive relation on $X \times Y$;

(M3') If $\{M(x_\alpha)\}_{\alpha \in A}$ is a decreasing net with respect to $\preceq$, i.e.

$$\alpha, \beta \in A, \alpha > \beta \implies \forall y_\beta \in M_Y(x_\beta), \exists y_\alpha \in M_Y(x_\alpha) : (x_\alpha, y_\alpha) \preceq (x_\beta, y_\beta)$$

and the net $\{x_\alpha\}_{\alpha \in A}$ converges to $x \in X$, then

$$\forall \alpha \in A : M(x) \preceq M(x_\alpha);$$

(M4') If $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M$ is a decreasing sequence with respect to $\preceq$, then $\{x_n\}_{n \in \mathbb{N}}$ is asymptotic.

Then, for each $x_0 \in X$ with $M_Y(x_0) \neq \emptyset$, there exists $\bar{x} \in X$ such that $M_Y(\bar{x}) \neq \emptyset$ and

1. $M(\bar{x}) \preceq M(x_0)$
2. $M(x) \preceq M(\bar{x}) \implies x = \bar{x}.$
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Proof. We define a binary relation on $X$ by setting

$$x' \preceq_X x \iff M(x') \preceq M(x)$$

in order to apply Theorem 24. With the help of (6.1), one can see that $\preceq_X$ is reflexive and transitive. (MP3) gives the lower closedness of $\preceq_X$. It remains to show the regularity. This can be done in the same way as the regularity for $\preceq$ in the proof of Theorem 21. Finally, a straightforward application of Theorem 24 yields (i) and (ii).

Note that the closedness assumption (M3') cannot be formulated merely in terms of the order relation on $X \times Y$ whereas the regularity assumption (M4') can. This is due to the fact that closedness in uniform spaces requires nets whereas regularity involves only sequences. Compare the proof of Theorem 21.

The corresponding result for $\preceq$ reads as follows.

Theorem 28 Let the following assumptions be satisfied:

(M1') $(X, U)$ is a uniform space and $X, Y$ as well as $M \subseteq X \times Y$ are nonempty sets;

(M2') $\preceq$ is a quasiorder, i.e., a reflexive and transitive relation on $X \times Y$;

(M3') If $\{M(x_\alpha)\}_{\alpha \in A} \subseteq M$ is an increasing net with respect to $\preceq$, i.e.

$$\alpha, \beta \in A, \alpha \succ \beta \implies \forall y_\beta \in M(x_\beta) \exists y_\alpha \in M(x_\alpha) : (x_\beta, y_\beta) \preceq (x_\alpha, y_\alpha)$$

and the net $\{x_\alpha\}_{\alpha \in A}$ converges to $x \in X$, then

$$\forall \alpha \in A : M(x_\alpha) \preceq M(x) ;$$

(M4') If $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M$ is a increasing sequence with respect to $\preceq$, then $\{x_n\}_{n \in \mathbb{N}}$ is asymptotic.

Then, for each $x_0 \in X$ with $M_Y(x_0) \neq \emptyset$, there exists $x \in X$ such that $M_Y(x) \neq \emptyset$ and

$$\begin{align*}
(i) & \quad M(x_0) \preceq M(x) \\
(ii) & \quad M(x) \preceq M(x) \implies x = \bar{x}.
\end{align*}$$

Proof. The proof is an application of Theorem 27 using the same arguments as in the proof of Theorem 22 applying Theorem 21.

Remark 27 As in the metric case, one can consider the special $Y = \{y_S\}$, a singleton. In this case, Theorem 27 reduces to Theorem 24 (as well as Theorem 28 to a maximal element reformulation of Theorem 24). On the other hand, Theorem 27 (as well as Theorem 28) are proven using Theorem 24 without any reference to the constructions in the proof of Theorem 24, especially not to Zorn’s lemma. In this sense, the theorems are equivalent.
6.1.5 Special cases of Theorem 24

In this section, we shall show that the fundamental lemma of Phelps (Lemma 1 of [101]) as well as its generalizations of Brønsted (Theorem 1 of [8]) and Mizoguchi (the lemma in [86]) are special cases of Theorem 24.

To begin with, we reformulate Brønsted’s theorem.

**Corollary 38** Let the following assumptions be satisfied:

(A1) \((X, \mathcal{U})\) is a complete uniform space;

(A2) \(\preceq\) is a quasiorder on \(X\) with lower closed section \(S(x) = \{x' \in X : x' \preceq x\}\);

(A3) The function \(f : X \to \mathbb{R} \cup \{+\infty\}\) is bounded below and monotone with respect to \(\preceq\), i.e.,
\[
x_1 \preceq x_2 \implies f(x_1) \leq f(x_2);
\]

(A4) For each \(E \in \mathcal{U}\), there is \(\delta > 0\) such that \(x_1 \preceq x_2\) and \(f(x_2) - f(x_1) < \delta\) implies \((x_1, x_2) \in E\).

Then, for each \(x_0 \in X\) with \(f(x_0) \in \mathbb{R}\), there is \(\bar{x} \in X\) such that
\[
\bar{x} \in S(x_0) \quad \text{and} \quad \{\bar{x}\} = S(\bar{x}).
\]

**Proof.** It suffices to verify the regularity of \(\preceq\) in order to apply Theorem 24. Take a decreasing sequence \(\{x_n\}_{n \in \mathbb{N}}\), i.e.,
\[
\forall n \in \mathbb{N} : x_{n+1} \preceq x_n.
\]

Fix \(E \in \mathcal{U}\) and take \(\delta > 0\) from assumption (A4). Since \(f\) is monotone and bounded below, the sequence \(\{f(x_n)\}_{n \in \mathbb{N}}\) is convergent. Hence, there is \(n_\delta \in \mathbb{N}\) such that
\[
\forall n \geq n_\delta : f(x_n) - f(x_{n+1}) < \delta.
\]

Assumption (A4) implies \((x_n, x_{n+1}) \in E\) for all \(n \geq n_\delta\), hence \(\{x_n\}_{n \in \mathbb{N}}\) is asymptotic. This proves the regularity of \(\preceq\). The assertions of the theorem follow from those of Theorem 24.

The original lemma of Phelps (Lemma 1 in [101]) is a consequence of Corollary 38. The details are not repeated here and can be found in [8].

**Corollary 39** Let the following assumptions be satisfied:

(A1) \((X, \mathcal{U})\) is a uniform space and \(\{p_\lambda\}_{\lambda \in \Lambda}\) a family of pseudometrics generating the uniformity;

(A2) \(\preceq\) is a quasiorder on \(X\) with lower closed section \(S(x) = \{x' \in X : x' \preceq x\}\);

(A3) \(\{f_\lambda\}_{\lambda \in \Lambda}\) is a family of functions \(f_\lambda : X \to \mathbb{R}\) such that each \(f_\lambda\) is bounded below on \(X\) and monotone with respect to \(\preceq\), i.e.
\[
\forall \lambda \in \Lambda : (x_1 \preceq x_2 \implies f_\lambda(x_1) \leq f_\lambda(x_2)).
\]
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(A4) For each \( \lambda \in \Lambda \) and each \( \varepsilon > 0 \), there is \( \delta_\lambda > 0 \) such that \( x_1 \preceq x_2 \) and \( f_\lambda (x_2) - f_\lambda (x_1) < \delta_\lambda \) implies \( p_\lambda (x_1, x_2) < \varepsilon \).

Then, for each \( x_0 \in X \), there is \( \bar{x} \in X \) such that

\[
\bar{x} \in S(x_0) \quad \text{and} \quad \{\bar{x}\} = S(\bar{x}).
\]

**Proof.** Again, it suffices to verify the regularity of \( \preceq \) in order to apply Theorem 24. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a decreasing sequence with respect to \( \preceq \). Repeating the arguments from the proof of Corollary 38 with \( f \) replaced by \( f_\lambda \), we obtain

\[
\forall \lambda \in \Lambda : \exists n_\lambda : \forall n \geq n_\lambda : p_\lambda (x_{n+1}, x_n) < \varepsilon.
\]

Since the sets

\[
E_{\lambda, \varepsilon} = \{(x, x') \in X \times X : p_\lambda (x, x') < \varepsilon\}, \lambda \in \Lambda, \varepsilon > 0
\]

form a base of the uniformity, the sequence \( \{x_n\}_{n \in \mathbb{N}} \) is asymptotic. Hence Theorem 24 can be applied to finish the proof. \( \blacksquare \)

In view of Proposition 33, the above corollary can be formulated replacing the family of pseudometrics by a family of quasimetrics. Finally, we give a formulation with a function \( f : X \to Y \), \( (Y, \preceq, \circ) \) a normally ordered, topological Abelian group. If \( X \) is separated uniform, such a group exists and additionally an order metric \( D : X \times X \to Y \) generating the uniform structure and the topology on \( X \), cf. Section 2.2.3.

To formulate the result, an additional condition is needed. Let \( X, Y \) as above such that the following condition is satisfied:

\( (R) \) Every sequence \( \{y_n\}_{n \in \mathbb{N}} \), that is decreasing with respect to \( \preceq \) and bounded from below, is asymptotic, i.e.,

\[
\forall B \in \mathcal{B}(\theta), \exists n_B \in \mathbb{N}, \forall n \geq n_B : y_n \circ (y_{n+1})^{-1} \in B,
\]

where \( \mathcal{B}(\theta) \) is a neighborhood base of \( \theta \in Y \) consisting of full sets.

**Corollary 40** Let the following assumptions be satisfied:

(A1) \( (X, \mathcal{U}) \) is a complete uniform space, \( (Y, \circ, \preceq) \) is a normally ordered, topological Abelian group satisfying condition \( (R) \) above;

(A2) \( \preceq \) is a quasiorder on \( X \) with lower closed sections \( S(x) = \{x' \in X : x' \preceq x\} \);

(A3) The function \( f : X \to Y \) is bounded below and monotone with respect to \( \preceq \), i.e.,

\[
x_1 \preceq x_2 \implies f(x_1) \leq f(x_2);
\]

(A4) For all \( E \in \mathcal{U} \) there is \( B \in \mathcal{B}(\theta) \) such that

\[
x_1 \preceq x_2, f(x_2) \circ (f(x_1))^{-1} \in B \implies (x_1, x_2) \in E.
\]

Then, for each \( x_0 \in X \), there is \( \bar{x} \in X \) such that

\[
\bar{x} \in S(x_0) \quad \text{and} \quad \{\bar{x}\} = S(\bar{x}).
\]
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Proof. Again, the only thing to check is the regularity of $\preceq$. Take a decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$. Then, the sequence $\{f(x_n)\}_{n \in \mathbb{N}} \subseteq Y$ is decreasing with respect to $\preceq$. Take $E \in \mathcal{U}$ and consider $B \in \mathcal{B}(\theta)$ from (A4). There is $n_B \in \mathbb{N}$ such that
\[
\forall n \geq n_B : x_{n+1} \preceq x_n, \quad f(x_n) \circ (f(x_{n+1}))^{-1} \in B,
\]
since $\{f(x_n)\}_{n \in \mathbb{N}}$ is asymptotic according to (R). (A4) implies $(x_n, x_{n+1}) \in E$ for all $n \geq n_B$. Hence $\{x_n\}_{n \in \mathbb{N}}$ is asymptotic as desired and we may apply Theorem 24 to obtain the assertions of the corollary. □

Of course, Corollary 40 is a generalization of Corollary 38. A related result is Theorem 5 of [124].

6.2 Results with functions into ordered monoids

6.2.1 Ekeland’s principle over quasiordered monoids

It is possible to give a uniform space formulation of all results of Section 4.2. We pick out three main theorems to show the principal procedure at work, namely Ekeland’s principle and its equilibrium version as well as Caristi’s fixed point theorem.

We start with Ekeland’s principle for functions mapping a uniform space into a quasiordered monoid. The first result of this type for extended real valued function on uniform spaces can be found in [8]. Therein, Brønsted proved a common generalization of Ekeland’s theorem [30], Theorem 1.1, and Lemma 1 of [101] due to Phelps playing in linear topological spaces. Mizoguchi [86] gave a slight generalization of Brønsted’s results as well as a fixed point theorem of Kirk–Caristi type for uniform spaces and a drop theorem in locally convex spaces. Moreover, she established the equivalence of these results. In several papers [89], [91], [90], [92], [93], Nemeth generalized Ekeland’s principle to functions mapping a uniform space into an ordered topological Abelian group. Also, Khanh [73] dealt with functions mapping so called L-spaces into ordered linear spaces. Finally, in [51] set valued variants of Ekeland’s principle and fixed point theorems for uniform spaces have been proven.

The following results involve order premetrics on uniform spaces in the sense of Definition 30.

Corollary 41 Let the following assumptions be satisfied:
(A1) $(X, \mathcal{U})$ is a complete uniform space and $(Y, \circ, \leq)$ a quasiordered monoid;
(A2) $\Phi : X \times X \to Y$ is an order premetric;
(A3) The function $f : X \to Y$ and $\tilde{y} \in Y$, $x_0 \in X$ are such that
   (i) $\tilde{y} \leq f(x)$ for all $x \in X$;
   (ii) $\Phi$ is regular with respect to $\tilde{y}, f(x_0) \in Y$;
(A4) If the net $\{x_\alpha\}_{\alpha \in A} \subseteq X$ converges to $x \in X$ and
\[
\forall \alpha, \beta \in A, \quad \alpha \succ \beta : f(x_\alpha) \circ \Phi(x_\alpha, x_\beta) \leq f(x_\beta),
\]
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then \( f(x) \circ \Phi(x, x) \leq f(x) \) for all \( \alpha \in A \).

Then, there is \( \bar{x} \in X \) such that

\[
\begin{align*}
(i) & \quad f(\bar{x}) \circ \Phi(\bar{x}, x_0) \leq f(x_0) \\
(ii) & \quad f(x) \circ \Phi(x, \bar{x}) \leq f(\bar{x}) \quad \implies \quad x = \bar{x}.
\end{align*}
\]

**Proof.** The proof is by checking the assumptions of Theorem 24 for the relation \( x' \preceq x \):

\( \preceq \) is reflexive since \( \leq \) is reflexive and (P1) of Definition 30 holds. It is transitive by (P2) and the transitivity of \( \leq \). (M4) follows directly from assumption (A4).

It remains to check the regularity of \( \preceq \). Let \( \{x_n\}_{n \in \mathbb{N}} \subseteq X \) be such that \( x_{n+1} \preceq x_n \) for all \( n \in \mathbb{N} \), i.e.,

\[
f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \leq f(x_n).
\]

The transitivity of \( \preceq \) implies

\[
f(x_{n+1}) \circ \Phi(x_{n+1}, x_n) \circ \Phi(x_n, x_{n-1}) \leq f(x_n) \circ \Phi(x_n, x_{n-1}) \leq f(x_{n-1}).
\]

Continuing this process, we obtain for each \( n \in \mathbb{N} \)

\[
f(x_{n+1}) \circ \sum_{k=0}^{n} \Phi(x_{k+1}, x_k) \leq f(x_0).
\]

Since \( \tilde{y} \leq f(x_m) \) for each \( m \in \mathbb{N} \) by (A2), it follows

\[
\tilde{y} \circ \sum_{k=0}^{n} \Phi(x_{k+1}, x_k) \leq f(x_0).
\]

Since by (A3) \( \Phi \) is regular with respect to \( \tilde{y}, f(x_0) \), the sequence \( \{x_n\}_{n \in \mathbb{N}} \) is asymptotic. Applying Theorem 24 yields the desired result. \( \blacksquare \)

We consider a set valued mapping \( T : X \rightarrow \hat{P}(X) \) in order to prove a fixed point theorem of Kirk–Caristi type.

**Corollary 42** Let the following assumptions be satisfied:

Let (A1) to (A4) of Corollary 41 be in force. If the mapping \( T : X \rightarrow \hat{P}(X) \) satisfies the weak contraction condition

\[
\forall x \in X, \exists x' \in T(x) : f(x') \circ \Phi(x', x) \leq f(x), \quad \text{(WC)}
\]

then \( T \) has a fixed point, i.e., there is \( \bar{x} \in X \) such that \( \bar{x} \in T(\bar{x}) \).

If the mapping \( T : X \rightarrow \mathcal{P}(X) \) satisfies the strong contraction condition

\[
\forall x \in X, \forall x' \in T(x) : f(x') \circ \Phi(x', x) \leq f(x), \quad \text{(SC)}
\]

then \( T \) has an invariant point, i.e., there is \( \bar{x} \in X \) such that \( \{\bar{x}\} = T(\bar{x}) \).
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Proof. Use Corollary 41 instead of Corollary 9 in the proof of Corollary 10.

Conversely, Corollary 41 can be proven using the fixed point result above. To see this, one has to proceed along the same lines as in the metric case. Compare the remarks after Corollary 10.

The Theorems 6 and 8 of [124] are also fixed point theorems of Kirk–Caristi type on uniform spaces. They involve real valued functions and a family of pseudometrics generating the uniformity, respectively.

The next result deals with a function $F: X \times X \to Y$ instead of $f: X \to Y$.

Corollary 43 Let the following assumptions be satisfied:

(A1) $(X, \mathcal{U})$ is a complete uniform space and $(Y, \circ, \leq)$ a quasiordered monoid;
(A2) The function $F: X \times X \to Y$, $\tilde{y} \in Y$ and $x_0 \in X$ are such that
   (i) $F(x_1, x_3) \leq F(x_1, x_2) \circ F(x_2, x_3)$ for all $x_1, x_2, x_3 \in X$;
   (ii) $\tilde{y} \leq F(x_0, x)$ for all $x \in X$;
(A3) $\Phi: X \times X \to Y$ is a regular order premetric with respect to $\tilde{y}, \theta \in Y$;
(A4) If the net $\{x_\alpha\}_{\alpha \in A} \subseteq X$ converges to $x \in X$ and
   
   \begin{align*}
   \forall \alpha, \beta \in A, \alpha \succ \beta: & \quad F(x_\alpha, x_\beta) \circ \Phi(x_\beta, x_\alpha) \leq \theta, \\
   \text{then } & \quad F(x_\alpha, x) \circ \Phi(x, x_\alpha) \leq \theta \text{ for all } \alpha \in A.
   \end{align*}

Then, there exists $\bar{x} \in X$ such that

\begin{align*}
\text{(i) } & \quad F(x_0, \bar{x}) \circ \Phi(\bar{x}, x_0) \leq \theta \\
\text{(ii) } & \quad F(\bar{x}, x) \circ \Phi(x, \bar{x}) \leq \theta \implies x = \bar{x}.
\end{align*}

First Proof. We check the assumptions of Theorem 16 for the relation

$x' \preceq x : \iff x' = x \text{ or } F(x, x') \circ \Phi(x', x) \leq \theta.$

being reflexive and transitive by the properties of $\Phi$, $F$ and $\leq$. (M4) follows directly from assumption (A4). The regularity of $\preceq$ can be seen in the same way as in the proof of Corollary 13. Therefore, we may apply Theorem 24 obtaining the desired result.

We shall indicate a sufficient condition for (A4) of Corollary 41. A function $f: X \to Y$ is called lower monotone iff for each net $\{x_\alpha\}_{\alpha \in A} \subseteq X$ converging to some $x \in X$ and satisfying $x_\alpha \preceq x_\beta$ for $\alpha \succ \beta$ the inequality $f(x) \leq f(x_\alpha)$ holds true for all $\alpha \in A$. Compare [93] for this kind of condition. It can be interpreted as a generalization of lower semicontinuity.

Moreover, an order premetric $\Phi: X \times X \to Y$ is called lower monotone with respect to the first variable iff for each net $\{x_\alpha\}_{\alpha \in A} \subseteq X$ converging to $x \in X$ and satisfying $x_\alpha \preceq x_\beta$ for $\alpha \succ \beta$ we have for all $x' \in X$

\begin{align*}
y_1, y_2 \in Y, \forall \alpha \in A: \quad & y_1 \circ \Phi(x_\alpha, x') \leq y_2 \implies y_1 \circ \Phi(x, x') \leq y_2.
\end{align*}
Lemma 8 Let $X$, $Y$ be as in Corollary 41, the function $f : X \rightarrow Y$ be lower monotone and the order premetric $\Phi$ be lower monotone with respect to the first variable. Then (A4) of Corollary 41 is satisfied.

Proof. Take a net $\{x_{\alpha}\}_{\alpha \in A} \subseteq X$ converging to $x \in X$ such that

$$\forall \alpha, \beta \in A, \alpha \succ \beta : f(x_{\alpha}) \circ \Phi(x_{\alpha}, x_{\beta}) \leq f(x_{\beta}).$$

Then, since $\theta \leq \Phi(x_{\alpha}, x_{\beta}),$

$$f(x_{\alpha}) \leq f(x_{\alpha}) \circ \Phi(x_{\alpha}, x_{\beta}) \leq f(x_{\beta})$$

and therefore $f(x_{\alpha}) \leq f(x_{\beta})$ for all $\alpha \succ \beta$ since $\leq$ is transitive. The lower monotonicity of $f$ implies $f(x) \leq f(x_{\alpha})$ for all $\alpha \in A$. For $\alpha, \beta \in A, \alpha \succ \beta$ we obtain

$$f(x) \circ \Phi(x_{\alpha}, x_{\beta}) \leq f(x_{\alpha}) \circ \Phi(x_{\alpha}, x_{\beta}) \leq f(x_{\beta}).$$

Since $\Phi$ is lower monotone with respect to the first variable, this implies

$$f(x) \circ \Phi(x, x_{\beta}) \leq f(x_{\beta})$$

as desired.

6.2.2 Power sets of quasiordered monoids

This subsection contains results parallel to those of Section 1.3.1.

Corollary 44 Let the following assumptions be satisfied:

(A1) $(X, \mathcal{U})$ is a complete uniform space, $(Y, \circ, \leq)$ an ordered monoid and $(\mathcal{Y}, \circ, \preceq)$ the ordered monoid generated by $\mathcal{Y} := \mathcal{P}(Y)$;

(A2) The function $f : X \rightarrow Y$ and $W \in \mathcal{Y}$ are such that

$$\forall x \in X : W \preceq f(x);$$

(A3) $\Phi : X \times X \rightarrow \mathcal{Y}$ is a regular order premetric with respect to $W, f(x_{0}) \in \mathcal{Y};$

(A4) If the net $\{x_{\alpha}\}_{\alpha \in A} \subseteq X$ converges to $x \in X$ and

$$\forall \alpha, \beta \in A, \alpha \succ \beta : f(x_{\alpha}) \circ \Phi(x_{\alpha}, x_{\beta}) \preceq f(x_{\beta}),$$

then $f(x) \circ \Phi(x_{\alpha}, x) \preceq f(x_{\alpha})$ for all $\alpha \in A.$

Then, there exists $\bar{x} \in X$ such that

(i) $f(\bar{x}) \circ \Phi(\bar{x}, x_{0}) \preceq f(x_{0})$

(ii) $x \in X, f(x) \circ \Phi(x, \bar{x}) \preceq f(\bar{x}) \implies x = \bar{x}.
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Proof. By Theorem 11, \((\mathcal{P}(Y), \odot, \preceq)\) is a quasiordered monoid. Defining the relation

\[ x' \preceq x \iff f(x') \odot \Phi(x', x) \preceq f(x) \]

on \(X\), the assumptions of Corollary 41 are easy to check. Its conclusions yield the desired result.

Corollary 45 Let the assumptions of Corollary 44 be satisfied with \(\preceq\) replaced by \(\preceq\). Then, there exists \(\bar{x} \in X\) such that

\begin{itemize}
  \item[(i)] \(f(\bar{x}) \odot \Phi(\bar{x}, x_0) \preceq f(x_0)\)
  \item[(ii)] \(x \in X, f(x) \odot \Phi(x, \bar{x}) \preceq f(\bar{x}) \implies x = \bar{x}\).
\end{itemize}

Proof. Replace \(\preceq\) by \(\preceq\) in the proof of Corollary 44.

6.2.3 Single valued functions

We show that Nemeth’s results in [93] follow from Theorem 24. They involve a function \(f\) mapping a uniform space into a topological Abelian group. Compare Corollary 46.

In the following corollary, \(Y\) is an ordered group not order complete in general. As usual, we can adjoin a largest element \(y_l\) obtaining an ordered monoid.

Corollary 46 Let the following assumptions be satisfied: (A1) \((X, U)\) is a complete uniform space and \((Y, \circ, \leq)\) an ordered topological Abelian group;
(A2) The function \(f : X \to Y \cup \{y_l\}\) and \(\tilde{y} \in Y\) are such that \(\tilde{y} \leq f(x)\) for all \(x \in X\);
(A3) \(\Phi : X \times X \to Y\) is a regular order premetric with respect to \(\tilde{y}\), \(f(x_0)\) for \(x_0 \in X\);
(A4) If the net \(\{x_\alpha\}_{\alpha \in A} \subseteq X\) converges to \(x \in X\) and

\[\forall \alpha, \beta \in A, \alpha \succ \beta : f(x_\alpha) \odot \Phi(x_\alpha, x_\beta) \leq f(x_\beta),\]

then \(f(x) \odot \Phi(x, x_\alpha) \leq f(x_\alpha)\) for all \(\alpha \in A\).

Then, there is \(\bar{x} \in X\) such that

\begin{itemize}
  \item[(i)] \(f(\bar{x}) \odot \Phi(\bar{x}, x_0) \leq f(x_0)\)
  \item[(ii)] \(f(x) \odot \Phi(x, \bar{x}) \leq f(\bar{x}) \implies x = \bar{x}\).
\end{itemize}

Proof. Apply Corollary 41 to \((Y \cup \{y_l\}, \circ, \leq)\).

Taking \(Y = V\) a topological linear space and corresponding order premetrics we are led to further results parallel to those of Section 4.3, 4.4 and 4.5. The results of Section 4.7 also have counterparts for a uniform space \(X\). Let us mention that Theorem 1 of Brønsted in [8] is also a special case of Corollary 46.

Similar results with a family of quasimetrics instead of an order premetric can be found in [49].
6.3 A partial minimal element theorem on complete uniform spaces

Let $(X, \mathcal{U})$ be a complete uniform space and $Y$ a nonempty set. The goal is to extend Theorem 24 to order relations on $X \times Y$. The result is parallel to Theorem 23. Let $\preceq$ be a quasiordering on $X \times Y$. A net $\{(x_\alpha, y_\alpha)\}_{\alpha \in A} \subseteq X \times Y$ is said to be decreasing iff

$$\forall \alpha, \beta \in A, \alpha \succ \beta : (x_\alpha, y_\alpha) \preceq (x_\beta, y_\beta).$$

A quasiorder on $X \times Y$ is called regular on $M \subseteq X \times Y$ iff for each decreasing sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M$ the sequence $\{x_n\}_{n \in \mathbb{N}}$ is asymptotic, i.e.,

$$\forall E \in \mathcal{U}, \exists n_E \in \mathbb{N}, \forall n \geq n_E : (x_{n+1}, x_n) \in E.$$

**Remark 28** Let $(X, \mathcal{U})$ be a uniform space and $Y$ a nonempty set. A regular quasiordering $\preceq$ on $X \times Y$ is partially antisymmetric. To see this, proceed as in Remark 24.

We state a result for complete uniform spaces parallel to Theorem 23.

**Theorem 29** Let the following assumptions be satisfied:

1. $(X, \mathcal{U})$ is a complete uniform space, $Y$ and $M \subseteq X \times Y$ are nonempty sets;
2. $\preceq$ is a quasiordering on $X \times Y$;
3. The quasiorder $\preceq$ is regular on $M$;
4. If $\{(x_\alpha, y_\alpha)\}_{\alpha \in A} \subseteq M$ is a decreasing net such that $\{x_\alpha\}_{\alpha \in A}$ converges to $x \in X$, then there exists $y \in Y$ such that $(x, y) \in M$ and

$$\forall \alpha \in A : (x, y) \preceq (x_\alpha, y_\alpha).$$

Then, for each $(x_0, y_0) \in M$, there exists $(\bar{x}, \bar{y}) \in M$ such that

1. $(\bar{x}, \bar{y}) \preceq (x_0, y_0)$
2. $(x, y) \in M$, $(x, y) \preceq (\bar{x}, \bar{y}) \implies x = \bar{x}.$

**Proof.** Consider the section $S(x_0, y_0) := \{(x, y) \in M : (x, y) \preceq (x_0, y_0)\}$. Let $S_0 \subseteq S(x_0, y_0)$ be a totally ordered subset of $S(x_0, y_0)$, namely a decreasing net $\{(x_\alpha, y_\alpha)\}_{\alpha \in A}$ with some directed index set $A$. Then $\{x_\alpha\}_{\alpha \in A}$ is a Cauchy net by (MP3). To see this, one can argue in the same way as in the proof of Theorem 23. By completeness, $\{x_\alpha\}_{\alpha \in A}$ is convergent to some $\hat{x} \in X$, hence there is $\hat{y} \in Y$ such that

$$\forall \alpha \in A : (\hat{x}, \hat{y}) \preceq (x_\alpha, y_\alpha).$$

This shows that $S_0$ is bounded below in $S(x_0, y_0)$. Zorn’s lemma, applied to the partially antisymmetric quasiorder $\preceq$, ensures the existence of a partial minimal point $(\bar{x}, \bar{y}) \in S(x_0, y_0)$. This completes the proof.

Using Theorem 29 we may obtain results which are the analogues to those of Chapter 5 in complete uniform space. We do not go into the details, but switch to the case of sequentially complete uniform spaces.