Two observations gave rise to the developments of this chapter. First, there are two proofs for the central result in Brønsted’s paper [8], the first one involves Zorn’s lemma, the second one does not, but only a countable induction argument. The question arises, in which cases the countable induction argument is sufficient. Secondly, minimal element theorems on sequentially complete uniform spaces have been established by A. Löehne and the author in [51] using a scalarization technique and the Brézis–Browder theorem [6]. The proof of the latter involves a countable induction argument only. Again, the question is under which assumptions this is sufficient. The results of this chapter show that in presence of a monotone real valued function with suitable properties linking the order and the uniform structure it is not necessary to use full versions of Zorn’s lemma. Since only sequences are involved, the completeness assumption of the last chapter can be weakened to sequential completeness.

7.1 The basic theorem with sequential completeness

7.1.1 Preliminaries

Let $(X,\mathcal{U})$ be uniform space. The quasiorner $\leq$ on $X$ is called sequentially lower closed iff the section $S(x) = \{x' \in X : x' \leq x\}$ is sequentially lower closed, i.e., if $\{x_n\}_{n \in \mathbb{N}} \subseteq S(x)$ is decreasing with respect to $\leq$ and convergent to $\bar{x} \in X$, then $\bar{x} \in S(x)$.

7.1.2 The basic theorem

Theorem 30 Let the following assumptions be satisfied:

(M1) $(X,\mathcal{U})$ is a separated, sequentially complete uniform space;

(M2) $\leq$ is a quasiorner on $X$;
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(M3) The function $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper, bounded from below on $X$ and monotone with respect to $\preceq$, i.e.

$$x_1 \preceq x_2 \implies f(x_1) \leq f(x_2),$$

moreover, for each $E \in \mathcal{U}$, there is $\delta > 0$ such that

$$x_1 \preceq x_2, \quad f(x_2) - f(x_1) < \delta \implies (x_1, x_2) \in E;$$

(M4) The quasiorder $\preceq$ is sequentially lower closed.

Then, for each $x_0 \in \text{dom } f$ there exists $\bar{x} \in \text{dom } f$ such that

$$\bar{x} \in S(x_0) \quad \text{and} \quad \{\bar{x}\} = S(\bar{x}).$$

PROOF. Starting with $x_0$ we choose a sequence according to

$$x_{n+1} \in S(x_n), \quad f(x_{n+1}) \leq \inf_{x \in S(x_n)} f(x) + \frac{1}{n}.$$  

The monotonicity of $f$ implies $f(x_{n+1}) \leq f(x_n)$ for all $n \in \mathbb{N}$. Since $f$ is bounded from below, the sequence $\{f(x_n)\}_{n \in \mathbb{N}}$ converges to some $r \in \mathbb{R}$.

Take $E \in \mathcal{U}$ and choose $n_E \in \mathbb{N}$ such that $f(x_{n_E}) < r + \delta$ with $\delta$ from (M3). This implies for $n \geq n_E$

$$f(x_{n_E}) - f(x_n) < r + \delta - r = \delta.$$  

Hence, for all $m \geq n \geq n_E$

$$f(x_m) - f(x_n) < \delta$$

holds true implying $(x_n, x_m) \in E$. Therefore, $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Since $X$ is sequentially complete, it converges to some $\bar{x} \in X$ and by (M4), $\bar{x} \in S(x_n)$ for each $n \in \mathbb{N}$. Especially, $\bar{x} \in S(x_0)$ holds true.

Let $x \preceq \bar{x}$, $x \neq \bar{x}$. Since $f$ is monotone, the yields $f(x) \leq f(\bar{x})$. On the other hand, the transitivity of $\preceq$ implies $x \preceq \bar{x} \preceq x_n$ for each $n \in \mathbb{N}$. The rules for the choice of $x_{n+1}$ yield

$$f(\bar{x}) \leq f(x_{n+1}) \leq f(x) + \frac{1}{n}.$$  

This yields $f(\bar{x}) \leq r \leq f(x)$, hence $f(\bar{x}) = r = f(x)$.

Again, take an arbitrary $E \in \mathcal{U}$. Since $\{f(x_n)\}_{n \in \mathbb{N}}$ converges to $f(x)$, there is $n_E \in \mathbb{N}$ such that

$$\forall n \geq n_E : f(x) - f(x_n) < \delta$$

This implies $(x_n, x) \in E$. Since $(X, \mathcal{U})$ is separated, we can conclude $x = \bar{x}$. 

\[\square\]
7.1. The basic theorem with sequential completeness

7.1.3 Equivalent formulations of the basic theorem

Without serious difficulties it is possible to transform the equivalent formulations of the basic minimal element theorem for metric spaces to the case of sequentially complete uniform spaces. We shall give the statements and refer for the details of the proofs to Section 4.1.3 and 7.1.3.

**Theorem 31** Let (M1) through (M4) of Theorem 30 be in force and, additionally, $T : X \to \mathcal{P}(X)$ be a set-valued mapping. If $T$ satisfies

$$\forall x \in X, \exists x' \in T(x) : x' \preceq x,$$

(WC)

then there is $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$, i.e., $\bar{x}$ is a fixed point of $T$. If $T$ satisfies

$$\forall x \in X, \forall x' \in T(x) : x' \preceq x,$$

(SC)

then there is $\bar{x} \in X$ such that $\{\bar{x}\} = T(\bar{x})$, i.e., $\bar{x}$ is an invariant point of $T$.

**Proof.** Each point $\bar{x}$ satisfying the conclusions of Theorem 30 does the job. \hfill $\blacksquare$

**Theorem 32** Let (M1) through (M4) of Theorem 30 be in force and, additionally:

(M5) The set $M \subseteq X \times Y$ satisfies

$$\forall x \in S(x_0) \setminus M \ \exists x' \in S(x) \setminus \{x\}.$$ 

Then, there exists $\bar{x} \in S(x_0) \cap M$.

**Proof.** By Theorem 30, there exists $\bar{x} \in S(x_0)$ such that $\{\bar{x}\} = S(\bar{x})$. By assumption (M5), $\bar{x} \in M$, hence $\bar{x} \in M \cap S(x_0)$.

\hfill $\blacksquare$

Again, Theorem 30 can be derived from Theorem 31 and Theorem 32 in the same way as Theorem 16 from Theorem 17 and Theorem 18, respectively.

7.1.4 Set relation ordering principle

In this section, the analogues to Theorems 21 and 22 shall be established.

Let $X, Y$ as well as $M \subseteq X \times Y$ be nonempty sets. Again, we define as in Section 4.1.6, $M(x) := \{(x', y) \in X \times Y : x' = x, (x', y) \in M\} \in \hat{\mathcal{P}}(X \times Y)$ and $M_Y(x) := \{y \in Y : (x, y) \in M\} \in \hat{\mathcal{P}}(Y)$ for $x \in X$.

Let $\preceq$ be a quasiorder on $M$. Then, $\{M(x) : x \in X\}, \preceq$ as well as $\{M(x) : x \in X\}, \preceq$ is quasiordered, compare Section 4.1.6.

**Theorem 33** Let the following assumptions be satisfied:

(M1') $(X, \mathcal{U})$ is a separated, sequentially complete uniform space and $Y$ as well as $M \subseteq X \times Y$ are nonempty sets;

(M2') $\preceq$ is a quasiorder on $X \times Y$;

...
(M3') The function $g : Y \to \mathbb{R} \cup \{+\infty\}$ is proper, bounded from below on $Y$ and satisfies the monotonicity condition

$$(x_1, y_1) \preceq (x_2, y_2) \implies g(y_1) \leq g(y_2),$$

moreover, for each $E \in \mathcal{U}$, there is $\delta > 0$ such that

$$(x_1, y_1) \preceq (x_2, y_2), \; g(y_2) - g(y_1) < \delta \implies (x_1, x_2) \in E;$$

(M4') If $\{x_n, y_n\}_{n \in \mathbb{N}} \subseteq M$ is a decreasing sequence with respect to $\preceq$ such that $\{x_n\}_{n \in \mathbb{N}}$ is convergent to $x \in X$, then there is $y \in Y$ such that $(x, y) \in M$ and

$$\forall n \in \mathbb{N} : (x, y) \preceq (x_n, y_n).$$

Then, for each $x_0 \in X$ with $M_Y(x_0) \cap \text{dom} \; g \neq \emptyset$, there exists $\bar{x} \in X$ such that $M_Y(\bar{x}) \neq \emptyset$ and

1. $M(\bar{x}) \preceq M(x_0)$
2. $M(x) \preceq M(\bar{x}) \implies x = \bar{x}.$

PROOF. Define a binary relation on $X$ by setting

$$x' \preceq_X x \iff M(x') \preceq M(x).$$

in order to apply Theorem 30. From the definition of $\preceq$, one can see that $\preceq_X$ is reflexive and transitive.

To check (M3) of Theorem 30, define a function $f : X \to \mathbb{R} \cup \{+\infty\}$ by

$$f(x) := \inf_{y \in M_Y(x)} g(y).$$

Then $f$ is proper and bounded below on $X$ since $g$ is proper and bounded below on $Y$. Moreover, $(x, y) \in M$, $y \in \text{dom} \; g$ implies $x \in \text{dom} \; f$. Assume that $x_1 \preceq_X x_2$, i.e., $M(x_1) \preceq M(x_2)$. The definition of $\preceq$ and the monotonicity property of $g$ yield

$$\forall y_2 \in M_Y(x_2) \exists y_1 \in M_Y(x_1) : g(y_1) \leq g(y_2)$$

implying $f(x_1) \leq f(x_2)$, i.e., $f$ is monotone with respect to $\preceq_X$. Fix $E \in \mathcal{U}$ and take $\delta > 0$ from (M3'). Then there is $y_2 \in M_Y(x_2)$ such that $g(y_2) \leq f(x_2) + \frac{\delta}{2}$ according to the definition of $f$. Since for each $y_1 \in M_Y(x_1)$ we have $f(x_1) \leq g(y_1)$, this implies

$$g(y_2) - g(y_1) \leq f(x_2) - f(x_1) + \frac{\delta}{2}.$$

Hence from $x_1 \preceq_X x_2$, $f(x_2) - f(x_1) < \frac{\delta}{2}$ we may conclude $(x_1, x_2) \in E$.

It remains to show the sequential lower closedness of $\preceq_X$. This is straightforward by taking a $\preceq_X$-decreasing and converging sequence and construct a $\preceq$-decreasing sequence in the same way as in the proof of Theorem 21. Then (M4') is sufficient for (M4) of Theorem 30.

Finally, an application of the latter theorem yields the desired results.

Again, the corresponding result for $\preceq$ has to be formulated as a maximal element result for the same reasons as Theorem 22.
Theorem 34 Let the following assumptions be satisfied:

(M1') \((X, \mathcal{U})\) is a separated, sequentially complete uniform space and \(Y\) as well as \(M \subseteq X \times Y\) are nonempty sets;

(M2') \(\preceq\) is a quasiorder on \(X \times Y\);

(M3') The function \(g : Y \to \mathbb{R} \cup \{+\infty\}\) is proper, bounded from below on \(Y\) and satisfies the monotonicity condition

\[(x_1, y_1) \preceq (x_2, y_2) \implies g(y_1) \geq g(y_2),\]

moreover, for each \(E \in \mathcal{U}\), there is \(\delta > 0\) such that

\[(x_1, y_1) \preceq (x_2, y_2), \ g(y_1) - g(y_2) < \delta \implies (x_1, x_2) \in E;\]

\(M4')\) If \(\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M\) is a increasing sequence with respect to \(\preceq\) such that \(\{x_n\}_{n \in \mathbb{N}}\) is converges to \(x \in X\), then there is \(y \in Y\) such that \((x, y) \in M\) and

\[\forall n \in \mathbb{N} : (x_n, y_n) \preceq (x, y).\]

Then, for each \(x_0 \in X\) with \(M_Y(x_0) \cap \text{dom } g \neq \emptyset\), there exists \(\bar{x} \in X\) such that \(M_Y(\bar{x}) \neq \emptyset\) and

\[(i) \quad M(x_0) \preceq M(\bar{x})\]

\[(ii) \quad M(\bar{x}) \preceq M(x) \implies x = \bar{x}.\]

Proof. The proof is an application of Theorem 33 using the same arguments as in the proof of Theorem 22 applying Theorem 21. Thus, the relation \(\preceq'\) and \(\preceq'\) are defined by

\[(x_1, y_1) \preceq' (x_2, y_2) \iff (x_2, y_2) \preceq (x_1, y_1)\]

and

\[M(x_1) \preceq' M(x_2) \iff M(x_2) \preceq M(x_1),\]

respectively. Assumption (M3) of Theorem 33 is satisfied for \(g\) and \(\preceq'\). For more details compare the proof of Theorem 22.

Of course, assumption (M3') of Theorem 34 can be formulated using a function \(g : Y \to \mathbb{R} \cup \{-\infty\}\) being bounded from above and satisfying

\[(x_1, y_1) \preceq (x_2, y_2) \implies g(y_1) \leq g(y_2).\]

Then, one has to define \(g' := -g\) using \((-1)(-\infty) = +\infty\) in order to apply Theorem 33.

7.2 The basic theorem on a product space

As in Chapter 5 for the case of a metric space \(X\) we establish a partial minimal element theorem for subsets \(M \subseteq X \times Y\).
Theorem 35 Let the following assumptions be satisfied:

(A1) \((X, U)\) is a sequentially complete, separated uniform space; \(Y\) is a nonempty set and \(M \subseteq X \times Y\) is also nonempty;

(A2) \(\preceq\) is a quasiorder on \(X \times Y\);

(A3) The function \(f : Y \to \mathbb{R} \cup \{+\infty\}\) is bounded from below on

\[
Y_M := \{ y \in Y : \exists x \in X : (x, y) \in M \}
\]

and monotone with respect to \(\preceq\), i.e.,

\[(x_1, y_1) \preceq (x_2, y_2) \implies f(y_1) \leq f(y_2),\]

moreover, for each \(E \in U\), there is \(\delta > 0\) such that

\[(x_1, y_1) \preceq (x_2, y_2), \; f(y_2) - f(y_1) < \delta \implies (x_1, x_2) \in E;\]

(A4) If the sequence \(\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M\) is decreasing with respect to \(\preceq\) and \(\{x_n\}_{n \in \mathbb{N}}\) converges to some \(x \in X\), then there is \(y \in Y\) such that \((x, y) \in M\) and \((x, y) \preceq (x_n, y_n)\) for each \(n \in \mathbb{N}\).

Then, for each \((x_0, y_0) \in M\) with \(f(y_0) \in \mathbb{R}\), there exists \((\bar{x}, \bar{y}) \in M\) such that

(i) \((\bar{x}, \bar{y}) \preceq (x_0, y_0)\)

(ii) \((x, y) \in M, \; (x, y) \preceq (\bar{x}, \bar{y}) \implies x = \bar{x}, \; f(y) = f(\bar{y}).\)

If, additionally, the assumption

(A5) \((x_1, y_1) \preceq (x_2, y_2), \; y_1 \neq y_2\) implies \(f(y_1) < f(y_2)\);

holds true, then \((\bar{x}, \bar{y})\) can be chosen to be a minimal point of \(M\) with respect to \(\preceq\), i.e.,

\[
\{(\bar{x}, \bar{y})\} = S(\bar{x}, \bar{y}) \cap M.
\]

Proof. Starting with \((x_0, y_0)\) we choose a sequence according to

\[
(x_{n+1}, y_{n+1}) \in S(x_n, y_n) \cap M, \quad f(y_{n+1}) \leq \inf_{(x, y) \in S(x_n, y_n) \cap M} f(y) + \frac{1}{n}.
\]

The monotonicity of \(f\) implies \(f(y_{n+1}) \leq f(y_n)\) for all \(n \in \mathbb{N}\). Since \(f\) is bounded from below on \(Y_M\), the sequence \(\{f(y_n)\}_{n \in \mathbb{N}}\) converges to some \(r \in \mathbb{R}\).

Take \(E \in U\) and choose \(n_E \in \mathbb{N}\) such that \(f(y_{n_E}) < r + \delta\) with \(\delta\) from (A3). This implies for \(n \geq n_E\)

\[
f(y_{n_E}) - f(y_n) < r + \delta - r = \delta.
\]

Hence, for all \(m \geq n \geq n_E\)

\[
f(y_m) - f(y_n) < \delta
\]

holds true implying \((x_n, x_m) \in E\). Therefore, \(\{x_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence converging to some \(\bar{x} \in X\) by completeness. By (A4), there is \(\bar{y} \in Y\) such that \((\bar{x}, \bar{y}) \in M\) and

\[
\forall n \in \mathbb{N}: (\bar{x}, \bar{y}) \preceq (x_n, y_n).
\]
7.2. The basic theorem on a product space

Let \((x, y) \preceq (\bar{x}, \bar{y})\). The monotonicity property of \(f\) implies \(f(y) \leq f(\bar{y})\). On the other hand, the transitivity of \(\preceq\) implies \((x, y) \preceq (\bar{x}, \bar{y}) \preceq (x_n, y_n)\) for each \(n \in \mathbb{N}\). The rules for the choice of \(y_{n+1}\) gives

\[ f(\bar{y}) \leq f(y_{n+1}) \leq f(y) + \frac{1}{n}. \]

This yields \(f(\bar{y}) \leq r \leq f(y)\), hence \(f(\bar{y}) = r = f(y)\).

Again, take an arbitrary \(E \in \mathcal{U}\). Since \(\{f(y_n)\}_{n \in \mathbb{N}}\) converges to \(f(y)\), there is \(n \in \mathbb{N}\) such that

\[ \forall n \geq n_E : f(y) - f(y_n) < \delta \]

implying \((x_n, x) \in E\). Since \((X, \mathcal{U})\) is separated, we may conclude \(x = \bar{x}\). Since \(f(\bar{y}) = f(y)\), \((\bar{x}, \bar{y}) \in M\) is minimal with respect to \(\preceq\) if (A5) is satisfied. This completes the proof of the theorem.

Before investigating some special cases of Theorem 35, we mention a version where the order relation is defined in terms involving the function \(f\). Consider \(X, Y, M, \preceq, f\) as in Theorem 35 and define an order relation by

\[(x_1, y_1) \preceq_f (x_2, y_2) \iff \left\{ \begin{array}{l} (x_1, y_1) = (x_2, y_2) \text{ or} \\ (x_1, y_1) \preceq (x_2, y_2) \text{ and } f(y_1) < f(y_2). \end{array} \right. \]

Obviously, \((x_1, y_1) \preceq_f (x_2, y_2)\) implies \((x_1, y_1) \preceq (x_2, y_2)\). Therefore, it is easily seen that the assumptions (A1) to (A5) of Theorem 35 are satisfied for \(\preceq_f\) if (A1) to (A4) are satisfied for \(\preceq\). This is the idea of the proof of the following theorem.

**Theorem 36** Let the assumptions (A1) through (A4) of Theorem 35 be in force. Then, for each \((x_0, y_0) \in M\) with \(y_0 \in \text{dom}\ f\) there exists \((\bar{x}, \bar{y}) \in M\) such that

\[ (i) \quad (\bar{x}, \bar{y}) \preceq_f (x_0, y_0) \]

\[ (ii) \quad (x, y) \in M, \quad (x, y) \preceq_f (\bar{x}, \bar{y}) \quad \Rightarrow \quad (x, y) = (\bar{x}, \bar{y}), \]

i.e., \((\bar{x}, \bar{y})\) is a minimal point of \(M\) with respect to \(\preceq_f\).

**Proof.** According to the remarks above, an obvious application of Theorem 35.

Next, we produce a series of corollaries from Theorem 35 by special choices of \(Y\) and the order relation \(\preceq\). Thereby, many recent results can be proven, e.g. minimal point theorems from [47] and [44] as well as results from [51] and [50].

The first case involves a locally convex space \(Y\) and \(f\) is replaced by a continuous linear functional.

**Corollary 47** Let the following assumptions be satisfied:

(A1) \((X, \mathcal{U})\) is a sequentially complete, separated uniform space and \(\{p_\lambda\}_{\lambda \in \Lambda}\) a family of pseudometrics generating the uniformity; \(Y\) is a locally convex space and \(M \subseteq X \times Y\) is a nonempty set;
\((A2) \leq_K\) is a quasiorder on \(Y\) with \(K \subseteq Y\) being a convex set containing \(\theta \in Y\) and a cone in \(\mathcal{P}(Y)\), further, let \(k \in K \setminus \text{cl} K\); a relation \(\leq\) on \(X \times Y\) is defined via

\[(x_1, y_1) \leq (x_2, y_2) \iff \forall \lambda \in \Lambda : y_1 + p_\lambda (x_1, x_2) k \leq y_2;\]

\((A3)\) There is a bounded set \(W \subseteq Y\) such that

\[Y_M := \{y \in Y : \exists x \in X : (x, y) \in M\} \subseteq W \oplus K;\]

\((A4)\) If the sequence \(\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq M\) is decreasing with respect to \(\leq\) and \(\{x_n\}_{n \in \mathbb{N}}\) converges to some \(x \in X\), then there is \(y \in Y_M\) such that \((x, y) \leq (x_n, y_n)\) for each \(n \in \mathbb{N}\). Then, for each \((x_0, y_0) \in M\) there exists \((\bar{x}, \bar{y}) \in M\) such that

\[\begin{align*}
(i) & \quad (\bar{x}, \bar{y}) \leq (x_0, y_0) \\
(ii) & \quad (x, y) \in M, \quad (x, y) \leq (\bar{x}, \bar{y}) \quad \implies \quad x = \bar{x}.
\end{align*}\]

If, additionally,

\[(A5)\] \(K^+ := \{y^* \in Y^* : \forall y \in K \setminus \{0\} : y^* (y) > 0\} \neq \emptyset;\)

is satisfied, then \((\bar{x}, \bar{y})\) can be chosen to be a minimal point of \(M\) with respect to \(\leq\), i.e.

\[\{(\bar{x}, \bar{y})\} = S(\bar{x}, \bar{y}) \cap M.\]

**Proof.** Of course, \(\leq\) is a quasiorder. A standard separation argument, applied to \(-k\) and \(K\) yields a continuous linear functional \(y^* \in Y^*\) such that \(y^*(k) = 1\) and

\[\forall y \in K : y^* (y) \geq 0.\]

We set \(f(y) := y^* (y)\) and check assumption (M3) of Theorem 35. Since \(W\) is bounded and \(f\) nonnegative on \(K\), \(f\) is bounded below on \(W \oplus K\) and all the more on \(Y_M\). Take \(E \in \mathcal{U}\). Then there are \(r > 0\), \(\lambda \in \Lambda\) such that

\[E_{r, \lambda} := \{(x_1, x_2) \in X \times X : p_\lambda (x_1, x_2) < r\} \subseteq E\]

since the sets \(E_{r, \lambda}\) form a base of the uniformity \(\mathcal{U}\). If \((x_1, y_1) \leq (x_2, y_2)\) and \(f(y_2) - f(y_1) < r\), then

\[y^*(y_1) + p_\lambda (x_1, x_2) = y^*(y_1 + kp_\lambda (x_1, x_2)) \leq y^*(y_2)\]

since \(f\) is linear and nonnegative on \(K\). Therefore

\[p_\lambda (x_1, x_2) \leq y^*(y_2) - y^*(y_1) \leq f(y_2) - f(y_1) < r,\]

hence \((x_1, x_2) \in E_{r, \lambda} \subseteq E\) as desired.

Since assumption (A4) coincides with assumption (M4) of Theorem 35 we can apply the latter to get (i) and (ii). If, additionally, \(K^+ \neq \emptyset\), then simply take \(y^* \in K^+\) to ensure (M5). This completes the proof.  \(\blacksquare\)
Corollary 47 is a generalization of Theorem 1 in [47] and Theorem 3.10.4 of [44]. Replacing the relation \( \preceq \) by \( \preceq' \) in the sense of Theorem 36, we obtain generalizations of Theorem 4 in [47] and Theorem 3.10.7 of [44].

The generalizations mainly concern the space \( X \) and the boundedness assumption: We deal with sequentially complete uniform spaces instead of complete metric spaces. Moreover, a single element \( \hat{y} \) is replaced by a bounded set \( W \) in assumption (A2). Note that Corollary 47 has a counterpart using quasimetrics instead of pseudometrics. Also, a formulation with an order metric is possible.

The reach of Corollary 47 is limited by the appearance of a nontrivial continuous linear functional on \( Y \), i.e., as a rule, \( Y \) has to be a locally convex space. Moreover, it has been observed in [47], [44] as well as in [51] and [50] that the boundedness assumption can be relaxed by a weaker one. In the following corollary, a sublinear functional on a linear space being linear only on a one dimensional subspace is used as a substitute for the continuous linear functional in Corollary 47. This allows to deal with merely linear spaces and to replace the boundedness assumption by a weaker one. Functionals of this type have been introduced and investigated by C. Tammer and P. Weidner in [39], [40], [41] and extensively in [125]. Compare also Theorem 2.3.1. in [44].

**Corollary 48** Let the following assumptions be satisfied:

(A1) \((X,\mathcal{U})\) is a sequentially complete, separated uniform space and \( \{p_\lambda\}_{\lambda \in \Lambda} \) a family of pseudometrics generating the uniformity; \( Y \) is a linear space and \( M \subseteq X \times Y \) is a nonempty set;

(A2) \( \preceq_K \) is a quasiorder on \( Y \) with \( K \subseteq Y \) being a convex set containing \( \theta \in Y \) and a cone in \( \mathcal{P}(Y) \), further, let \( k \in K \setminus -K \); a relation \( \preceq \) on \( X \times Y \) is defined via

\[
(x_1, y_1) \preceq (x_2, y_2) \iff \forall \lambda \in \Lambda : y_1 + p_\lambda(x_1, x_2) k \leq_K y_2;
\]

(A3) There exist \( \hat{y} \in Y \) and \( \hat{t} \in \mathbb{R} \) such that

\[
\{ y \in Y : \exists x \in X : (x, y) \in M \} \cap \left( \{ \hat{y} - \hat{t}k \} \oplus (-K) \right) = \emptyset;
\]

(A4) If the sequence \( \{(x_n, y_n)\}_{n \in \mathbb{N}} \) is decreasing with respect to \( \preceq \) and \( \{x_n\}_{n \in \mathbb{N}} \) converges to some \( x \in X \), then there is \( y \in Y \) such that \( (x, y) \in M \) and \( (x, y) \preceq (x_n, y_n) \) for each \( n \in \mathbb{N} \).

Then, for each \( (x_0, y_0) \in M \) with \( y_0 \in \mathbb{R} \{k\} \oplus (-K) \) there exists \( (\bar{x}, \bar{y}) \in M \) such that

1. \( (\bar{x}, \bar{y}) \preceq (x_0, y_0) \)
2. \( (x, y) \in M, \ (x, y) \preceq (\bar{x}, \bar{y}) \implies x = \bar{x} \).

**Proof.** We are going to apply Theorem 35. Of course, (M1) and (M2) of this theorem are satisfied. We check (M3) using the function

\[
f(y) := \inf \{ t \in \mathbb{R} : y - \hat{y} \in \{tk\} \oplus (-K) \}.
\]
Since the function \( \varphi(y) := \inf \{ t \in \mathbb{R} : y \in \{ t k \} \oplus (-K) \} \) is monotone with respect to \( \leq_K \) and subadditive, \( f \) is monotone as well. Note that \( \varphi \) satisfies
\[
\forall s \in \mathbb{R}, \ y \in Y : \ \varphi(y + sk) = \varphi(y) + s.
\]
This property is called translation property, see [44], Section 2.3. Further, \( f \) is bounded below on \( Y_M := \{ y \in Y : \exists x \in X : (x, y) \in M \} \). To see this, assume the contrary, i.e., there is a \( \tilde{y} \in Y_M \) such that \( f(\tilde{y}) < -\tilde{t} \) and hence there is \( \tilde{t} \in \mathbb{R}, \ \tilde{t} < -\tilde{t} \) such that \( \tilde{y} - \tilde{y} \in \tilde{t} \{ k \} \oplus (-K) \). This implies
\[
\tilde{y} \in \{ \tilde{y} + \tilde{t} k \} \oplus (-K) = \{ \tilde{y} - \tilde{t} k \} \oplus \{ (\tilde{t} + \tilde{t}) k \} \oplus (-K) \subseteq \{ \tilde{y} - \tilde{t} k \} \oplus (-K)
\]
contradicting (A3). Hence \( f \) is bounded below on \( Y_M \).

Take \( E \in \mathcal{U} \) and \( r > 0, \lambda \in \Lambda \) such that \( E_{r, \lambda} \subseteq E \). Assuming \( (x_1, y_1) \preceq (x_2, y_2) \) and \( f(y_2) - f(y_1) < r \) we obtain by monotonicity and the translation property of \( \varphi \)
\[
p_\lambda(x_1, x_2) \preceq \varphi(y_2 - \tilde{y}) - \varphi(y_2 - \tilde{y}) = f(y_2) - f(y_1) < r,
\]
hence \( (x_1, x_2) \in E_{r, \lambda} \subseteq E \). Therefore, the assumptions of Theorem 35 are satisfied. Note that \( y_0 \in \mathbb{R} \{ k \} \oplus (-K) \) implies \( f(y_0) \in \mathbb{R} \).

The conclusions (i) and (ii) of Theorem 35 yield (i) and (ii) above. \( \blacksquare \)

Corollary 48 produces a generalization of Corollary 47: \( Y \) can be replaced by a linear space and the boundedness assumption can be weakened.

Note that within the setting of Corollary 48 it is difficult to give a sufficient condition for (A5) of Theorem 35, i.e., for the existence of a minimal point with respect to \( \preceq \).

Usually, topological properties are used as in part (g) of Theorem 2.3.1. in [44]. Note also that \( f(y) = f(\tilde{y}) \) can be added in (ii) of Corollary 48.

The set \( Y \) in Theorem 35 is arbitrary, hence the possibility of choosing \( Y \subseteq \mathcal{P}(V) \), \( V \) being a quasiordered linear space, is not excluded. We turn to this case in order to derive results similar to those of [50]. Since sets are compared, the order relations \( \succeq \) and \( \preceq \) appear.

**Corollary 49** Let the following assumptions be satisfied:

(A1) \( (X, \mathcal{U}) \) is a sequentially complete, separated uniform space and \( \{ p_\lambda \}_{\lambda \in \Lambda} \) a family of pseudometrics generating the uniformity; \( V \) is a linear space and \( \mathcal{M} \subseteq X \times \hat{\mathcal{P}}(V) \) is a nonempty set;

(A2) \( \preceq \) is a quasiorder on \( V \) with \( K \subseteq V \) being a convex set containing \( \theta \in V \) and a cone in \( \mathcal{P}(V) \) and \( k \in K \setminus -K \); a relation \( \preceq \) is defined via
\[
(x_1, W_1) \preceq (x_2, W_2) \iff \forall \lambda \in \Lambda : W_1 \oplus \{ p_\lambda(x_1, x_2) \} k \subseteq W_2;
\]

(A3) There exist \( \hat{\nu} \in V \) and \( \hat{t} \in \mathbb{R} \) such that
\[
\left( \bigcup_{(x, W) \in \mathcal{M}} W \right) \cap (\{ \hat{\nu} - \hat{t} k \} \oplus (-K)) = \emptyset;
\]
(A4) If the sequence \( \{ (x_n, W_n) \}_{n \in \mathbb{N}} \subseteq \mathcal{M} \) is decreasing with respect to \( \leq \) and \( \{ x_n \}_{n \in \mathbb{N}} \) converges to some \( x \in X \), then there is \( W \in \hat{\mathcal{P}}(V) \) such that \( (x, W) \in \mathcal{M} \) and \( (x, W) \preceq (x_n, W_n) \) for each \( n \in \mathbb{N} \).

Then, for each \( (x_0, W_0) \in \mathcal{M} \) with \( \mathbb{R} \{ k \} \oplus \{ \hat{v} \} \cap (W_0 \oplus K) \neq \emptyset \) there exists \( (\hat{x}, \hat{W}) \in \mathcal{M} \) such that

\[
\begin{align*}
(i) & \quad (\hat{x}, \hat{W}) \preceq (x_0, W_0) \\
(ii) & \quad (x, W) \in \mathcal{M}, \quad (x, W) \preceq (\hat{x}, \hat{W}) \quad \implies \quad x = \hat{x}.
\end{align*}
\]

**Proof.** Again, we wish to apply Theorem 35. It is not hard to verify that (M1) and (M2) of this theorem are matched as well as (M4). To verify (M3), we define a function \( f : \hat{\mathcal{P}}(V) \to \mathbb{R} \cup \{ \pm \infty \} \) by

\[
f(W) := \inf \{ t \in \mathbb{R} : tk + \hat{v} \in W \oplus K \}.
\]

Using this definition, the monotonicity property of \( f \) and the translation property

\[
\forall W \in \hat{\mathcal{P}}(V), \forall s \in \mathbb{R} : f(W \oplus \{sk\}) = f(W) + s
\]

can be proven straightforward. Let us show that \( f \) is bounded below on

\[
\left\{ W \in \hat{\mathcal{P}}(V) : \exists x \in X : (x, W) \in \mathcal{M} \right\}.
\]

Assume the contrary. Then there is \( (x, W) \in \mathcal{M} \) such that \( f(W) < -\hat{t} \). Hence there is \( s \in \mathbb{R}, s < \hat{t} \) such that

\[
sk + \hat{v} \in W \oplus K.
\]

Especially, \( W \neq \emptyset \). Take \( w \in W \). Then

\[
w \in \hat{v} + sk \oplus (-K) = \hat{v} + (s + \hat{t}) k - \hat{t}k \oplus (-K) \subseteq \hat{v} - \hat{t}k \oplus (-K)
\]

contradicting (A3). The last part of assumption (M3) can be proven as in the proof of Corollary 48. We may apply Theorem 35 to obtain (i) and (ii) above from its conclusions.

\[ \blacksquare \]

**Corollary 50** Let (A1) and (A2) of Corollary 49 be satisfied with \( \preceq \) replaced by \( \preceq \). Moreover, assume:

(A3) There exist \( \hat{v} \in V \) and \( \hat{t} \in \mathbb{R} \) such that

\[
(x, W) \in \mathcal{M} \implies W \notin \{ \hat{v} - \hat{t}k \} \oplus (-K)
\]

(A4) If the sequence \( \{ (x_n, W_n) \}_{n \in \mathbb{N}} \subseteq \mathcal{M} \) is decreasing with respect to \( \preceq \) and \( \{ x_n \}_{n \in \mathbb{N}} \) converges to some \( x \in X \), then there is \( W \in \hat{\mathcal{P}}(V) \) such that \( (x, W) \in \mathcal{M} \) and \( (x, W) \preceq (x_n, W_n) \) for each \( n \in \mathbb{N} \).

Then, for each \( (x_0, W_0) \in \mathcal{M} \) with \( W_0 \subseteq \{ t_0 k + \hat{v} \} \oplus (-K) \) for some \( t_0 \in \mathbb{R} \), there exists \( (\hat{x}, \hat{W}) \in \mathcal{M} \) such that

\[
\begin{align*}
(i) & \quad (\hat{x}, \hat{W}) \preceq (x_0, W_0) \\
(ii) & \quad (x, W) \in \mathcal{M}, \quad (x, W) \preceq (\hat{x}, \hat{W}) \quad \implies \quad x = \hat{x}.
\end{align*}
\]
Proof. As in the proof of Corollary 50, the only problem is to verify (M3) of Theorem 35 involving the function \( f: \hat{P}(V) \rightarrow \mathbb{R} \cup \{\pm \infty\} \) defined by

\[
f(W) := \inf \{ t \in \mathbb{R} : W \subseteq \{tk + \hat{v}\} \oplus (-K) \}.
\]

Using this definition, the monotonicity property of \( f \) and the translation property

\[
\forall W \in \hat{P}(V), \forall s \in \mathbb{R} : f(W \oplus \{sk\}) = f(W) + s
\]

can be proven straightforward. Let us show that \( f \) is bounded below on

\[
\left\{ W \in \hat{P}(V) : \exists x \in X : (x, W) \in \mathcal{M} \right\}.
\]

Assume the contrary. Then there is \((x, W) \in \mathcal{M} \) such that \( f(W) < -\hat{t} \). Hence there is \( s \in \mathbb{R}, s < \hat{t} \) such that

\[
W \subseteq \{sk + \hat{v}\} \oplus (-K) = \{(s + \hat{k})k - \hat{v}k + \hat{v}\} \oplus (-K) \subseteq \{-\hat{t}k + \hat{v}\} \oplus (-K)
\]

contradicting (A3). The last part of assumption (M3) of Theorem 35 can be proven as in the proof of Corollary 48. We may apply Theorem 35 to obtain (i) and (ii) above from its conclusions.

Both of Corollary 49 and Corollary 50 imply Corollary 48 by setting

\[
\mathcal{M} = \{(x, \{v\}) : (x, v) \in M\}.
\]

Note the complete symmetry of the constructions in Corollary 49 and 50, respectively. Again, a sufficient condition for (A5) is a difficult task and requires additional topological assumptions such as compactness. We refer to [52].